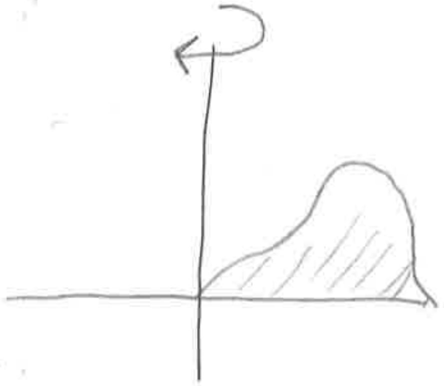
§6.2 - Volumes using Cylinderical Shells

<u>Motivational Example</u>: Find the volume of the solid of revolution by rotating the region $y = 3x^2 - x^3$, $y=0$ about the y-axis.



<u>Difficulty</u>: Finding r_{out} and r_{in} since we need them to be functions in y (not in x!)

In that case, we will use the idea of cylinderical shell by placing a rectangle (with distance x from y-axis) in the region and we rotate it.

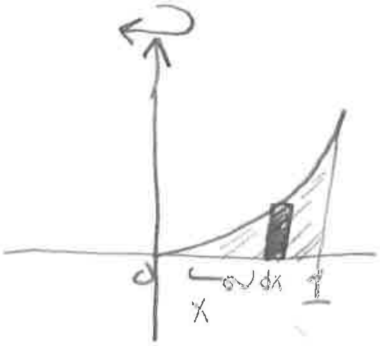
So the volume = (circumference) (height) (thickness)

$$= (2\pi \text{ (distance between the rectangle and axis of revolution)}) (\text{height})\, dx$$

So the total volume $= \int_a^b$ volume of the rectangle.

<u>Example 1</u>: Find the volume generated by rotating the region $y = x^2$, $y = 0$, $x = 1$ about y-axis.

<u>Solution</u>:

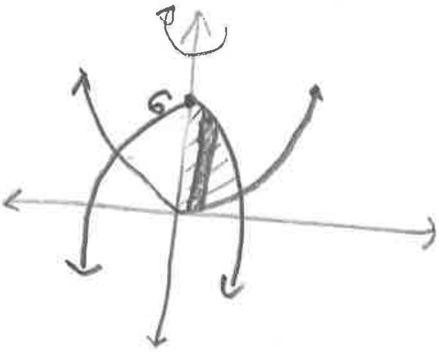


Volume of the rectangle $=(2\pi x)(x^2)\,dx$

$$\text{Volume} = \int_0^1 2\pi x^3\,dx = 2\pi\int_0^1 x^3\,dx$$

$$= 2\pi\left[\frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

$\boxed{2}$ $y = x^2$, $y = 6-2x^2$ about y- axis



$x^2 = 6-2x^2$

$3x^2 = 6$

$x^2 = \frac{6}{3} \Rightarrow x = \pm\sqrt{2}$

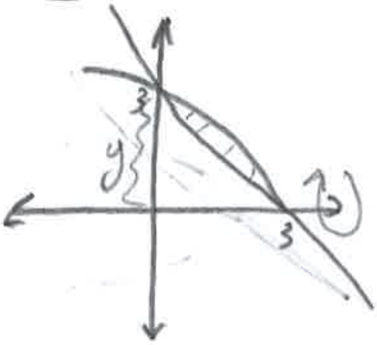
V of rectangle $= 2\pi(x)(6-2x^2-x^2)\,dx$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} 2\pi x(6-2x^2-x^2)\,dx$$

$\boxed{3}$ $y = \sqrt{x}$, $x = 0$, $y = 2$ about x-axist

$$V = 2\pi\int_0^2 y(y^2-0)\,dy = 8\pi$$

$\boxed{4}$ $x + y = 3$, $x = y - (y-1)^2$ about x-axis

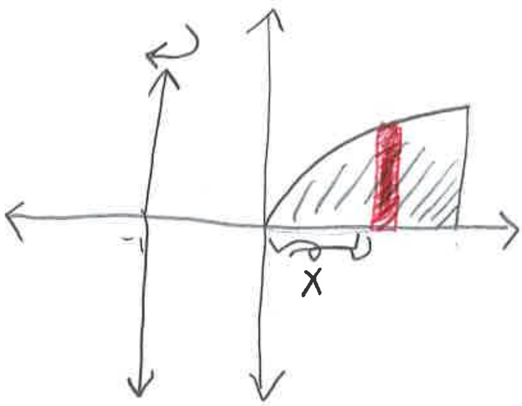


$$\text{Volume} = 2\pi(y)\big(y-(y-1)^2-(3-y)\big)dy$$

$$V = \int_0^3 2\pi y\big(y-(y-1)^2-3+y\big)\,dy$$

$$= \frac{27\pi}{2}$$

$\boxed{5}$ $y = \sqrt{x}$, $y = 0$, $x = 1$ about $x = -1$

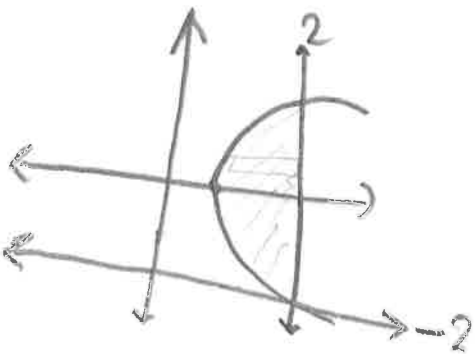


Volume of the rectangle

$$= \underbrace{\big(2\pi(x+1)\big)}_{\text{distance}}\underbrace{(\sqrt{x})}_{\text{height}}\underbrace{dx}_{\text{Thickness}}$$

$$\text{Total volume} = \int_0^1 2\pi(x+1)\sqrt{x}\,dx = \ldots$$

$\boxed{6}$ $\cancel{x} = y^2+1$, $x = 2$ about $y = -2$



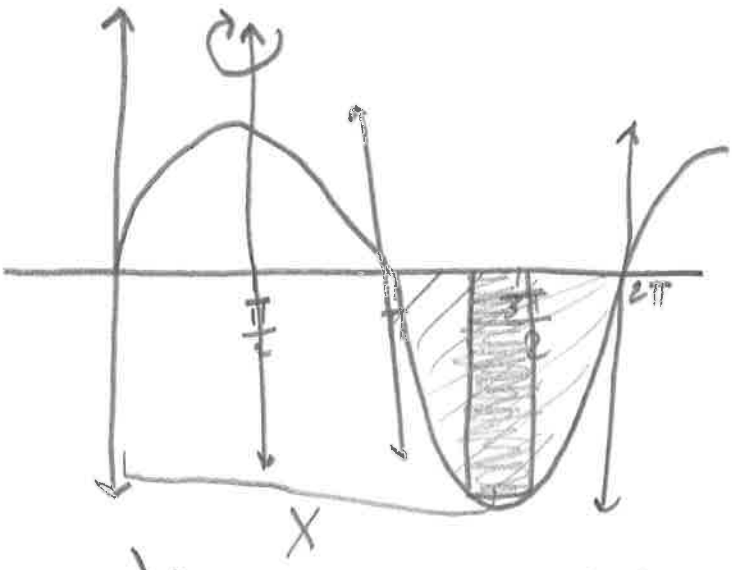
$h = 2 - y^2 - 1$

$y_1 = y + 2$

$$\int_{-1}^{1} 2\pi (y+2)(2-y^2-1)\,dy$$

$$= \frac{16\pi}{3}$$

$\boxed{7}$ $y = \sin x$, $y = 0$, $x = \pi$, $x = 2\pi$, about $\cancel{x} = \frac{\pi}{2}$

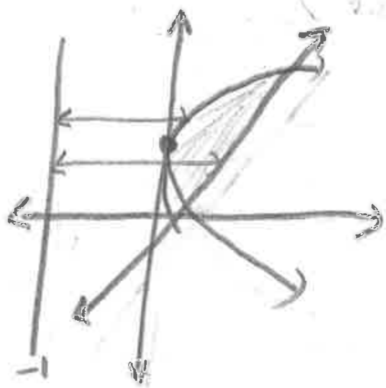


$$\text{Volume} = 2\pi\left(x - \frac{\pi}{2}\right)(0 - \sin x)$$

$$V = 2\pi \int_{\pi}^{2\pi} \left(x - \frac{\pi}{2}\right)(0 - \sin x)\,dx$$

Any method

8 $X = (y-1)^2$, $X - y = 1$ about $X = -1$

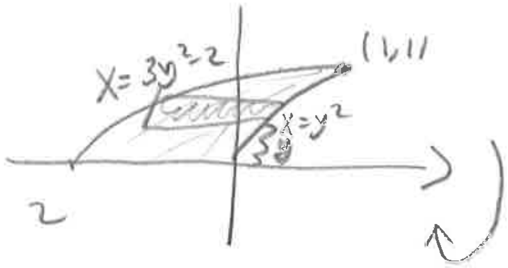


$$V = \pi \int_1^4 (r_{out})^2 - (r_{in})^2 \, dy$$

$$= \pi \int_1^4 (y+1)^2 - ((y-1)^2) \, dy$$

$$= 30\pi$$

9 Which method would you use to find the volume



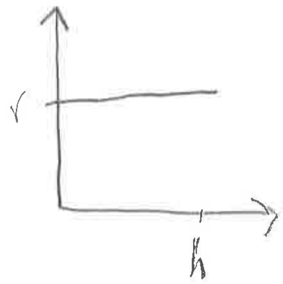
$$V = 2\pi \int_0^1 y(y^2 - (3y^2 - 2)) \, dy$$

10 $y = \sqrt{X}$, $y = 2$, $X = 0$ about $y = 2$ and $X = 4$.

$$V = 2\pi \int_0^4 (4-x)(2-\sqrt{x}) \, dX$$

Volumes of special Solids

1)



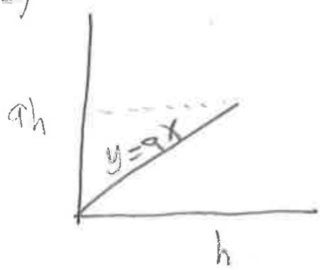
Circular cylinder of height h and radius r

$y = r$

$$V = \pi \int_0^h (r)^2 \, dx$$

$$V = \pi r^2 h$$

2)



Circular cone of height h and angle θ

$y = \tan\theta \, x$

$$V = \pi \int_0^h (\tan\theta \, x)^2 \, dx$$

$$= \pi (\tan\theta)^2 \frac{h^3}{3}$$

$$= \frac{1}{3} \pi h^3 (\tan\theta)^2$$

3)

Sphere of radius r

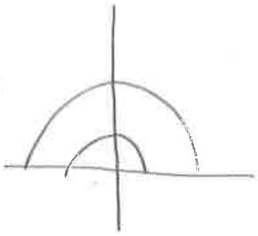
$y = \sqrt{r^2 - x^2}$

$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= 2\pi \left[r^3 + \frac{r^3}{3} \right] = \frac{4\pi r^3}{3}$$

4)

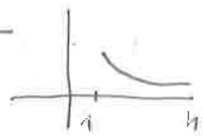


Torus

$r_{out} = \sqrt{r_{out}^2 - x^2}$

$r_{in} = \sqrt{r_{in}^2 - x^2}$

5) Horn $y = \frac{1}{x}$



$$V = \pi \int^h \left(\frac{1}{x}\right)^2 dx =$$