Chapter 8: Integration Techniques

§8.0 - Basic Approach and Substitution Method

Recall:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
- $\int e^x dx = e^x + c$
- $\int \frac{1}{x} dx = \ln x$
- $\int \sin x \, dx = -\cos x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$ (§7.5)

Substitution method:

Find $\int F(g(x)) \cdot g'(x) \, dx$

put $u = g(x) \rightsquigarrow \frac{du}{dx} = g'(x) \rightsquigarrow du = g'(x) \, dx$

$\int F(\underbrace{g(x)}_{u}) . \underbrace{g'(x) dx}_{du} = \int F(u) \, du$ and now we integrate.

Example 1:

$\boxed{1}$ $\int \frac{1}{3+2x} dx$, let $u = 3+2x \rightarrow du = 2dx \rightsquigarrow dx = \frac{1}{2} du$

$\int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln|3+2x| + c$

Example 2:
$$\int_{-1}^{2} \frac{dx}{3+2x} = \left[\frac{1}{2} \ln|3+2x| \right]_{-1}^{2} = \frac{1}{2}\ln 7 - \frac{1}{2}\ln 1$$
$$= \frac{1}{2}\ln 7$$

Exercise 1: $\int (6+8x)^5 \, dx$

Example 3:
$$\int \frac{1}{e^x + e^{-x}} dx = \int \frac{1}{e^x + \frac{1}{e^x}} dx = \int \frac{1}{\frac{e^{2x}+1}{e^x}} dx = \int \frac{e^x}{e^{2x}+1} dx$$

$$u = e^x \rightsquigarrow du = e^x dx$$

$$= \int \frac{1}{u^2+1} du = \tan^{-1} u + c$$
$$= \tan^{-1} e^x + c$$

Example 4: "Split up fractions"
$$\int \frac{x^3 + \sqrt{x}}{x^{3/2}} dx = \int \frac{x^3}{x^{\frac{3}{2}}} + \frac{x^{\frac{1}{2}}}{x^{\frac{3}{2}}} dx = \int x^{\frac{3}{2}} + x^{-1} dx$$
$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \ln x + c = \frac{2}{5} x^{\frac{5}{2}} + \ln x + c$$

Exercise 2:
$$\int \frac{x+1}{x-1} dx$$

Recall: (Double angle formula)

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x \longrightarrow \cos^2 x = \dfrac{1+\cos 2x}{2}$

$\searrow \sin^2 x = \dfrac{1-\cos 2x}{2}$

Example 5:

$$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2} \int 1 + \cos 2x \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} x + \frac{1}{4} \sin 2x + C$$

Exercise 3: $\int \sin^2 x \, dx$

Example 6: $\int \tan x \, dx = \int \dfrac{\sin x}{\cos x} dx$

$u = \cos x \rightsquigarrow du = -\sin x \, dx$

$$\int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

Example 7: Multiply by 1

$$\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} dx = \int \frac{1-\cos x}{1-\cos^2 x} dx$$

$$= \int \frac{1-\cos x}{\sin^2 x} dx = \int \csc^2 x - \csc x \tan x \, dx = -\cot x + \csc x + C$$

__Example 8:__ $\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}$

let $u = \sec x + \tan x \longrightarrow du = (\sec x \tan x + \sec^2 x)\,dx$

$= \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|\sec x + \tan x| + c$

__Exercise 5:__

a- Show $\int \frac{1}{\sqrt{x-x^2}}\,dx = \sin^{-1}(2x-1) + c$ using $u = 2x-1$

b- Show $\int \frac{1}{\sqrt{x-x^2}}\,dx = 2\sin^{-1}\sqrt{x} + c$ using $u = \sqrt{x}$

c- Show that $2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1) = \frac{\pi}{2}$ (hint: put $x = \frac{1}{2}$)