§8.2 Trigonometric Integrals

In this Section, we look at integrals of the form $\int \sin^3 x\,dx$, $\int \cos^n x\,dx$, $\int \sin^3 x \cos^7 x\,dx, \ldots$

<u>Recall:</u> $\sin^2 x = 1-\cos^2 x$, $\cos^2 x = 1-\sin^2 x$

<u>Example 1:</u> (or use integral by parts).

$$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx$$

$$= \int (\cos^2 x)^2 \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx$$

Put $u = \sin x \longrightarrow du = \cos x\,dx$

$$\int \cos^5 x\,dx = \int (1-u^2)^2\,du = \int 1-2u^2+u^4\,du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

<u>Example 2:</u>

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1-\cos 2x)^2\,dx = \frac{1}{4}\int 1-2\cos 2x+\cos^2 2x\,dx$$

$$= \frac{1}{4}\left(x - \sin 2x + \int \cos^2 2x\,dx\right)$$

$$= \frac{1}{4}\left(x - \sin 2x + \int \frac{1+\cos 4x}{2}\,dx\right)$$

$$= \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right) + C$$

Exercise 1: $\int \sin^3 x \, dx$.

Example 3: $\int \sin^3 x \cos^{-2} x \, dx = \int \sin^2 x \cos^{-2} x \cdot \sin x \, dx$

$= \int (1-\cos^2 x) \cos^{-2} x \cdot \sin x \, dx$

Put $u = \cos x \longrightarrow du = -\sin x \, dx$

$= -\int (1-u^2) u^{-2} \, du = -\int (u^{-2}-1) \, du = + u^{-1} - u + c$

$= \cos^{-1} x - \cos x + c$

Example 4:

$\int \sin^{2} x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$

$= \frac{1}{4}\int 1 - \cos^2 2x \, dx = \frac{1}{4}\int 1 - \frac{1+\cos 4x}{2} \, dx$

$= \frac{1}{4}\left(x - \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right) = \frac{1}{8}x - \frac{1}{32}\sin 4x$

Rule

$\int \sin^n x \cos^m x \, dx$
- if n or m odd, split that one
- if n & m even, use double-angle formula

Use integration by parts for $\int \sin^n x\,dx$, $\int \cos^n x\,dx$, $\int \tan^n x$, -

Example 5:- $\int \tan^3 x \sec^4 x\,dx$

$$= \int \tan^2 x \sec^2 x \cdot \tan x \sec x\,dx = \int (\sec^2 x - 1)\sec^2 x \cdot \tan x \sec x\,dx$$

Put $u = \sec x \rightarrow du = \sec x \tan x\,dx$

$$= \int (u^2-1)u^2\,du = \int u^4 - u^2\,du = \frac{u^5}{5} - \frac{u^3}{3} + c$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c.$$

Example 6:- $\int \tan^3 x \sec^4 x\,dx = \int \tan^3 x \sec^2 x \cdot \sec^2 x\,dx$

Put $u = \tan x \rightarrow du = \sec^2 x\,dx$

$$= \int u^3(1+u^2)\,du = \int u^3 + u^5\,du = \frac{u^4}{4} + \frac{u^6}{6} + c$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

Strategy

$\int \tan^m x \sec^n x\,dx$

n even $\longrightarrow$ split $\sec^2 x$

m odd $\longrightarrow$ split $\sec x \tan x$.

Exercise 4: $\int \tan^9 x \sec^4 x \, dx$