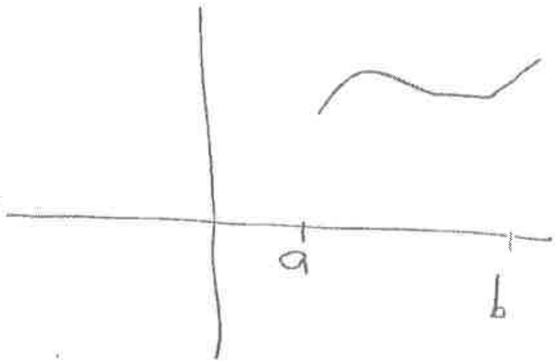
§ 8.3 Trigonometric Substitutions

1- The Arc-length:

Let $f(x)$ be a continuous differentiable function, the arc-length of $f(x)$ in the interval $a \le x \le b$ is given by

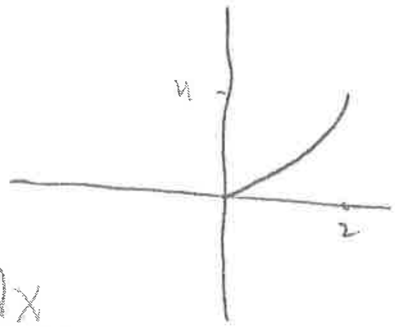


$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

Example 1: Find the arc-length of the parabola $y = x^2$ on $[0, 2]$

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

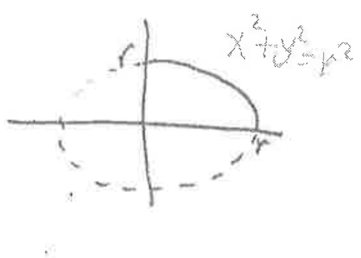
$$= \int_0^2 \sqrt{1+(2x)^2}\, dx = \int_0^2 \sqrt{1+4x^2}\, dx$$



Question: How do we find such integral?

Example 2: Find the circumference of a circle of radius r.

$C = 4 \cdot$ Arc-length of a quarter



$$C = 4 \cdot \int_0^r \sqrt{1+(y')^2}\, dx \quad , \quad y = \sqrt{r^2 - x^2}$$

$$y' = \frac{2x}{2\sqrt{r^2-x^2}} = \frac{x}{\sqrt{r^2-x^2}}$$

$$= 4 \cdot \int_0^r \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx$$

$$= 4 \cdot \int_0^r \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\, dx = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\, dx = 4r\int_0^r \frac{1}{\sqrt{r^2-x^2}}\, dx$$

Now we need to find integrals of the form $\sqrt{r^2-x^2}$.

2 - Integrals involving $a^2 - x^2$

use the substitution $x = a \sin\theta$

$$dx = a \cos\theta .$$

So $a^2 - x^2 = a^2 - (a\sin\theta)^2 = a^2 - a^2 \sin^2\theta$

$$= a^2(1-\sin^2\theta) = a^2 \cos^2\theta$$

Example: Find $\int \frac{1}{\sqrt{9-x^2}}\, dx$

Put $x = 3\sin\theta \rightarrow dx = 3\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{9-(3\sin\theta)^2}} \cdot 3\cos\theta\, d\theta = \int \frac{1}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta = \int \frac{3\cos\theta}{\sqrt{9}\cdot\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int 1\, d\theta = \theta + C$$

Now we want to return back to x.

$$x = 3\sin\theta \rightarrow \frac{x}{3} = \sin\theta \rightarrow \sin^{-1}\frac{x}{3} = \theta$$

$$\boxed{\int \frac{1}{\sqrt{9-x^2}}\, dx = \sin^{-1}\frac{x}{3} + C}$$

Exercise 1: Finish the integral of Example 2.

Example 4: $\int \frac{dx}{(16-x^2)^{\frac{3}{2}}}$

Put $x = 4\sin\theta \longrightarrow dx = 4\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{(16-x^2)^3}} = \int \frac{4\cos\theta}{\sqrt{(16-16\sin^2\theta)}^3}\, d\theta = \int \frac{4\cos\theta}{(4\cos\theta)^3}\, d\theta$$

$$= \int \frac{4\cos\theta}{4^3\cos\theta}\, d\theta = \frac{1}{16}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{16}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{16}(\tan\theta) + C = \frac{1}{16}\tan\theta + C.$$

Exercise 2: $\int \frac{\sqrt{9-x^2}}{x}\, dx$

Example 5: $\int x\sqrt{1-x^4}\,dx$

Put $u = x^2$, $du = 2x\,dx$

$$\int x\sqrt{1-x^4}\,dx = \int x\sqrt{1-u^2}\cdot\frac{du}{2x} = \frac{1}{2}\int\sqrt{1-u^2}\,du$$

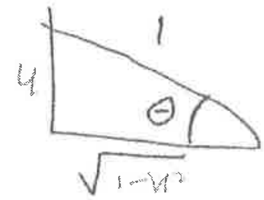
Now $u = \sin\theta \longrightarrow du = \cos\theta\,d\theta$

$$\frac{1}{2}\int\sqrt{1-u^2}\,du = \frac{1}{2}\int\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$= \frac{1}{2}\int\cos\theta\cdot\cos\theta\,d\theta = \frac{1}{2}\int\cos^2\theta\,d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right] = \frac{1}{4}\left[\theta + \frac{2}{2}\sin\theta\cos\theta\right] + c$$

$$= \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + c$$



$$= \frac{1}{4}\sin^{-1}u + \frac{1}{4}\cdot u\cdot\sqrt{1-u^2} + c$$

$$= \frac{1}{4}\sin^{-1}x^2 + \frac{1}{4}x^2\sqrt{1-x^4} + c.$$

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\sqrt{a^2-a^2\sin^2\theta} = a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$\sqrt{a^2+a^2\tan^2\theta} = a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sqrt{a^2\sec^2\theta-a^2} = a\tan\theta$

In general, for $\sqrt{a-bx^2}$, $x = \frac{\sqrt{a}}{\sqrt{b}}\sin\theta$.

Example 6: $\int \frac{x}{\sqrt{x^2-7}}\,dx$

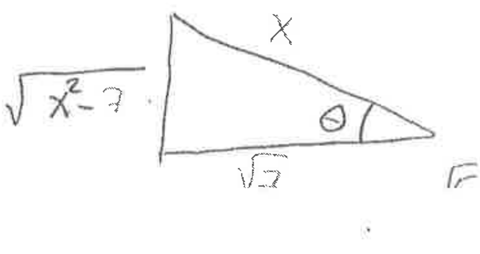
$x = \sqrt{7}\sec\theta \longrightarrow dx = \sqrt{7}\sec\theta\tan\theta\,d\theta$

$$\int \frac{x}{\sqrt{x^2-7}}\,dx = \int \frac{\sqrt{7}\sec\theta}{\sqrt{7\sec^2\theta-7}}\cdot\sqrt{7}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{7}\sec\theta}{\cancel{\sqrt{7}}\sqrt{\sec^2\theta-1}}\cdot\cancel{\sqrt{7}}\sec\theta\tan\theta\,d\theta = \int \frac{\sqrt{7}\sec^2\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}}$$

$$= \sqrt{7}\int \sec^2\theta\,d\theta = \sqrt{7}\tan\theta + c$$

$\left.\sec\theta = \frac{x}{\sqrt{7}}\right\} = \sqrt{7}\,\frac{\sqrt{x^2-7}}{\sqrt{7}} = \sqrt{x^2-7} + c$



Exercise 3 : Find the Arc-length of the parabola $y = x^2$ for $0 \leq x \leq 2$ as stated in example 1.

Exercise 4 : • Find $\int \frac{dx}{(1+x^2)^2}$

• $\int \frac{1}{x^2\sqrt{x^2+9}} \, dx$

• $\int \frac{x^2}{(x^2-1)^{5/2}} \, dx$

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• $\int \frac{e^t}{(1+e^{2t})^{3/2}} \, dt$