§ 8.4 - Partial Fractions

1- Polynomials :

Definition :

A <u>polynomial</u> over the real numbers is a sum

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad , \quad a_n, a_{n-1}, \dots, a_0 \in \mathbb{R}$$

$$= \sum_{k=0}^{n} a_k x^k$$

and n is called the <u>degree</u> of p.

Example 1 :

1. $p(x) = 3x^2 + 5x + 7$
2. $q(x) = x^4 + x + 1$
3. $p(x) = x^{11} - 1$
4. $q(x) = 3$

are all polynomials.

Definition :

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{j=0}^{m} b_j x^j$, then one defines

- $p(x) + q(x) = \sum_{k=0}^{\max(n,m)} (a_k + b_k) x^k$

- $p(x) q(x) = \sum_{k=0}^{m+n} c_k x^k \quad , \quad c_k = \sum_{i=0}^{k} a_i b_{k-i}$.

Example 2:

let $p(x) = x^2 + 4x$, $q(x) = x^3 + x$

- $p(x) + q(x) = x^3 + x^2 + 5x \cancel{+4}$
- $p(x)\,q(x) = (x^2 + 4x)(x^3 + x) = x^5 + x^3 + 4x^4 + 4x^2$

$$= x^5 + 4x^4 + x^3 + 4x^2.$$

Notation:

- polynomial of degree one is called linear.
- polynomial of degree two is called quadratic
- polynomial of degree 3 is called cubic.

2- Completing the Squares and important integrals

Recall:

$$x^2 + 2bx + b^2 = (x+b)^2.$$

To complete the square of

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c$$

$$= \left(x + \frac{b}{2}\right)^2 + c$$

Example 3: complete the square of the following.

1. $x^2 + 7x + 5 = x^2 + 7x + \frac{49}{4} - \frac{49}{4} + 5 = \left(x + \frac{7}{2}\right)^2 - \frac{29}{4}$.

2. $x^2+5 = x^2+5$

3. $x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = (x+\frac{1}{2})^2+\frac{3}{4}$

4. $x^2-3x = x^2-3x+\frac{9}{4}-\frac{9}{4} = (x-\frac{3}{2})^2-\frac{9}{4}$

Example 4: Find

(a) $\int \frac{1}{3x+7}\,dx$, (b) $\int \frac{1}{x^2+9}\,dx$, (c) $\int \frac{1}{x^2+x+1}\,dx$

(d) $\int \frac{x}{x^2+4}\,dx$

(a) $\int \frac{1}{3x+7}\,dx$, $u = 3x+7$

$du = 3\,dx \to dx = \frac{1}{3}du$

$$\int \frac{1}{u}\cdot\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u|+c = \frac{1}{3}\ln|3x+7|+c$$

(b) $\int \frac{1}{x^2+9}\,dx$, use $x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$$\int \frac{1}{9\tan^2\theta+9}\cdot 3\sec^2\theta\,d\theta = \int \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{1}{3}\int d\theta = \frac{1}{3}\theta+c$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

b) $\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+1}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

let $\quad x + \frac{1}{2} = \sqrt{\frac{3}{4}} \tan\theta$

$$dx = \frac{\sqrt{3}}{2} \sec^2\theta \, d\theta$$

$$\int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx = \int \frac{1}{\frac{3}{4}\tan^2\theta + \frac{3}{4}} \cdot \frac{\sqrt{3}}{2} \sec^2\theta \, d\theta$$

$$= \int \frac{\sqrt{\frac{3}{4}} \sec^2\theta}{\frac{3}{4}\sec^2\theta} d\theta = \frac{2\sqrt{3}}{3} \int 1 \, d\theta = \frac{2\sqrt{3}}{3} \theta + C$$

$$= \frac{2\sqrt{3}}{3} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right) + C.$$

(d) $\displaystyle\int \frac{x}{x^2+4} dx$

put $\quad u = x^2 + 4$

$du = 2x\,dx$

$$\int \frac{x}{x^2+4} dx = \int \frac{x}{u} \frac{1}{2x} du = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+4| + C.$$

Summary

- $\displaystyle\int \frac{1}{Ax+B} dx = \frac{1}{A}\ln|Ax+B|$ --- $u = Ax+B$.
- $\displaystyle\int \frac{1}{x^2+a^2} dx = \frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right)$ --- $x = a\tan\theta$
- $\displaystyle\int \frac{1}{x^2+bx+c} dx$ --- complete the square and $x + \frac{b^2}{4} = \sqrt{c - \frac{b^2}{4}}\tan$
- $\int \frac{x}{\ }$ --- $u = Ax^2 + B$

3- Long Division (Euclidean Division) on Polynomials

Example 5: perform long division on

$$
\begin{array}{r|l}
 & x + 1 \quad \leftarrow \text{Quotient} \\
\hline
x^2+2x+1 & x^3+3x^2+x+1 \\
 & -\,(\cancel{x^3} + 2x^2 + x) \\
\cline{2-2}
 & \cancel{x^2} + 1 \\
 & -\,(\cancel{x^2} + 2x + 1) \\
\cline{2-2}
 & -2x \quad \leftarrow \text{Reminder}
\end{array}
$$

we write

$$\frac{x^2+2x+1}{x^3+3x^2+x+1} = \underbrace{x+1}_{\text{Quotient}} + \frac{-2x}{x^2+2x+1} \leftarrow \text{Reminder}$$

Exercise 2:-

$$
\begin{array}{r|l}
\hline
x+3 & x^3+3x+x
\end{array}
$$

4- Methods of Partial Fractions

Goal:- To find integrals $\displaystyle\int \frac{p(x)}{q(x)}\,dx$ Polynomial

Step 1: If degree of $p(x) \geq$ degree of $q(x)$ perform long division on $\frac{p(x)}{q(x)}$ and work on the reminder part.

Step 2: Factor the denominator $q(x)$ into either linear factors $(AX-B)^i$ or irreducible quadratic polynomi-

$$\frac{1}{(ax^2+bx+c)^j} \quad (b^2-4ac<0).$$

Step 3: For each factor above, we write

$$\frac{r(x)}{(dx+e)^r(ax^2+bx+c)^s} = \frac{A_1}{(dx+e)} + \frac{A_2}{(dx+e)^2} + \cdots + \frac{A_r}{(dx+e)^r}$$

$$+ \frac{B_1x+C_1}{(ax^2+bx+c)} + \cdots + \frac{B_sx+C_s}{(ax^2+bx+c)^s}$$

Step 3: Integrate the partial fractions.

Example 6: Find $\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$

Step 2:

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{3x^2+7x-2}{x(x^2-x-2)} = \frac{3x^2+7x-2}{x(x-2)(x+1)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{x+1}.$$

Find A_1, A_2, A_3?

Multiply ~~both~~ both sides by $x(x-2)(x+1)$.

$$3x^2+7x-2 = A_1(x-2)(x+1) + A_2(x)(x+1) + A_3\,x(x-2)$$

Put $\underline{x=0}$:

$$-2 = A_1(0-2)(0+1) \longrightarrow -2 = -2A \longrightarrow \boxed{A_1 = 1}$$

Put $\underline{x=2}$:

$$24 = A_2(2)(2+1) \longrightarrow 24 = 6A_2 \longrightarrow \boxed{A_2 = 4}$$

Put $\underline{x=-1}$:

$$-6 = A_3(3) \longrightarrow \boxed{A_3 = -2}$$

So

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{1}{x} + \frac{4}{x-2} - \frac{2}{x+1}$$

Step 3 :

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx = \int \frac{1}{x}\,dx + \int \frac{4}{x-2}\,dx - \int \frac{2}{x+1}\,dx$$

difficult to integrate

Easy to integrate.

Example 7: Find $\int \frac{5x^2-3x+2}{x^3-2x^2}$.

Example 8: setting-up Partial Fractions

- $$\frac{x^2+1}{x^4-4x^3-32x^2} = \frac{x^2+1}{x^2(x^2-4x-32)} = \frac{x^2+1}{x^2(x+4)(x-8)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x+4} + \frac{C_1}{x-8}$$

- $$\frac{10}{(x-2)^2(x^2+2x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+2x+2}$$

Exercise 10:-

$$\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$$