§ 6.$\cancel{8}$7 Improper Integrals

Definition:

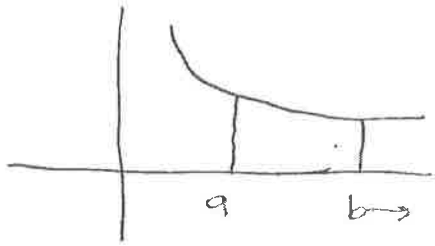
An integral $\int_a^b f(x)\,dx$ is called improper integral if

(1) $[a,b]$ is infinite interval ($a=-\infty$ or $b=\infty$)

(2) $f(x)$ is unbounded on $[a,b]$.

1 - Infinite intervals

- $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$



- $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$
- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$.

Recall
- $\frac{1}{\infty}=0$
- $e^{-\infty}=\frac{1}{e^{\infty}}=0$
- $e^{\infty}=\infty$

Example 1: Find the following integrals

(1) $\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b = \lim_{b\to\infty}\left[\frac{-1}{b}-\frac{-1}{1}\right]$

$\displaystyle = \lim_{b\to\infty}\left[\frac{-1}{b}+1\right] = 1.$

2) $\displaystyle\int_0^{\infty} e^{-3x}\,dx = \lim_{b\to\infty}\int_0^b e^{-3x}\,dx = \lim_{b\to\infty}\left[\frac{-1}{3}e^{-3x}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3b}+\frac{1}{3}e^{-3(0)}\right]$

$$= -\frac{1}{3}\underbrace{e^{-3\infty}}_{=0} + \frac{1}{3} = \frac{1}{3}.$$

(3) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{c} \frac{1}{1+x^2}\,dx + \int_{c}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{c} \frac{1}{1+x^2}\,dx + \lim_{b \to \infty} \int_{c}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \left[\tan^{-1} x\right]_a^c + \lim_{b \to \infty} \left[\tan^{-1} x\right]_c^b$$

$$= \lim_{a \to -\infty} \left[\tan^{-1} c - \tan^{-1} a\right] + \lim_{b \to \infty} \left[\tan^{-1} b - \tan^{-1} c\right]$$

$$= \tan^{-1} c - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1} c$$

$$= +\frac{\pi}{2} + \frac{\pi}{2} = \pi$$

<u>Exercise 1</u>: If $p > 1$, Find $\int_1^{\infty} \frac{1}{x^p}\,dx$

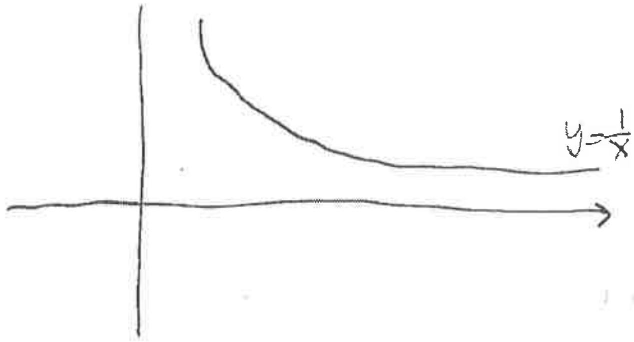
<u>Exercise 2</u>: If $0 < p < 1$, Find $\int_1^{\infty} \frac{1}{x^p}\,dx$

<u>Exercise 3</u>: If $p = 1$, Find $\int_1^{\infty} \frac{1}{x}\,dx$

<u>Summary</u>

$$\int_1^{\infty} \frac{1}{x^p}\,dx \quad \begin{cases} \text{Converges if } p > 1 \\ \text{Diverges if } p \le 1 \end{cases}$$

Homework 9: "Gabriel Horn"



(a) Find the volume of the surface by revolving $y=\frac{1}{x}$ around the x-axis in the interval $[1,\infty)$

(Hint: the formula of the volume is $V = 2\pi\int_a^b [f(x)]^2 dx$)

(b) Find the surface of the surface by revolving $y=\frac{1}{x}$ around the x-axis in the interval $[1,\infty)$

(Hint: The surface area is $S = 2\pi\int_1^b f(x)\sqrt{1+[f'(x)]^2}\,dx$)

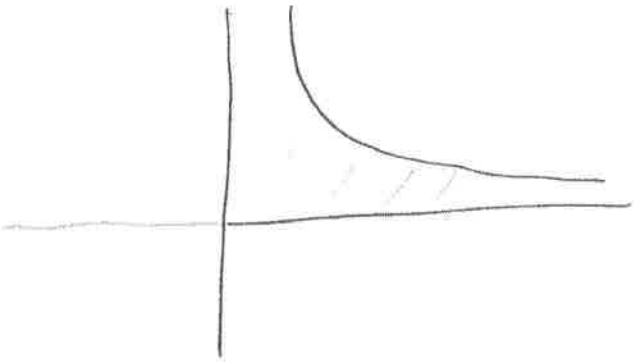
(c) Is there any contradiction between part (a) & (b)?

2 - Unbounded Integrands

Improper integral occurs if the function becomes infinite at some point in the interval

Example 2:

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper because $\frac{1}{\sqrt{x}}$ is infinite for $x \to 0$

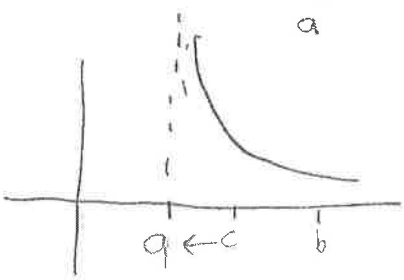


In this case, we write $\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a \to 0^+} \int_a^1 x^{-\frac{1}{2}}\,dx$

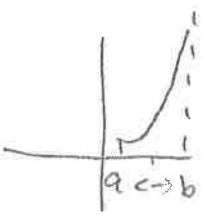
$= \lim_{a \to 0^+} \left[2x^{\frac{1}{2}} \right]_a^1 = \lim_{a \to 0^+} \left[2\sqrt{1} - 2\sqrt{a} \right] = 2.$

Rule 2: Improper integral for unbounded functions

1. if $\lim_{x \to a^+} f(x) = \pm\infty$, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

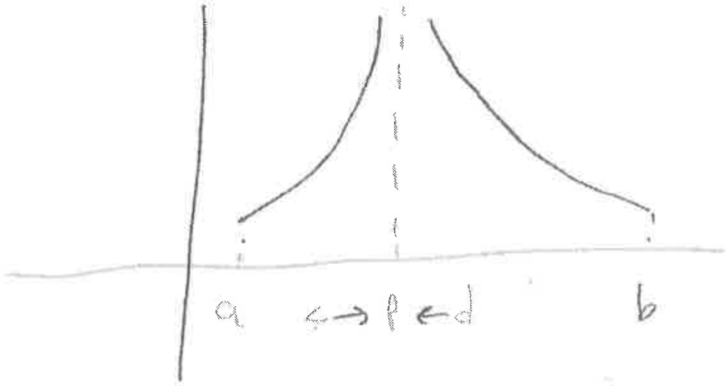


2. If $\lim_{x \to b^-} f(x) = \pm\infty$, then $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$



3. If f is unbounded at one point inside $[a,b]$, we have

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^a f(x)\,dx$$



Notation:

If the limit exist, we say the improper integral converges.

otherwise, we call it diverges.

Example 3: Find $\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx$

Note that $f(x)$ are unbounded at -3 & 3, so we break it at some point to get

two problems

$$\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx = \underbrace{\int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}\,dx}_{\text{improper integral}} + \underbrace{\int_{0}^{3} \frac{1}{\sqrt{9-x^2}}\,dx}_{\text{improper integral}}$$

$$= \lim_{c\to -3^+} \int_c^0 \frac{1}{\sqrt{9-x^2}}\,dx + \lim_{c\to 3^-} \int_0^c \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \lim_{c\to -3^+} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_c^0 + \lim_{c\to 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^c$$

$$= \left(\cancel{\sin^{-1}\left(\frac{0}{3}\right)} - \sin^{-1}\left(\frac{-3}{3}\right)\right) + \left(\sin^{-1}\left(\frac{3}{3}\right) - \cancel{\sin^{-1}\left(\frac{0}{3}\right)}\right)$$

$$= -\sin^{-1}(-1) + \sin^{-1}(1)$$

$$= -\frac{\pi}{2} + \frac{\pi}{2}$$

$$= 0$$

<u>Example 4</u>: $\displaystyle\int_1^{10} \frac{1}{(x-2)^{\frac{1}{3}}}\,dx$

$f(x)$ is unbounded at $x=2$, so

$$\int_1^{10} \frac{1}{(x-2)^{\frac{1}{3}}}\,dx = \int_1^{2} \frac{1}{(x-2)^{\frac{1}{3}}}\,dx + \int_2^{10} \frac{1}{(x-2)^{\frac{1}{3}}}\,dx$$

$$= \lim_{c\to 2^-} \int_1^{c} \frac{1}{(x-2)^{\frac{1}{3}}}\,dx + \lim_{c\to 2^+} \int_c^{10} \frac{1}{(x-2)^{\frac{1}{3}}}\,dx$$

$$= \lim_{c\to 2^-} \left[\frac{3}{2}(x-2)^{\frac{2}{3}}\right]_1^c + \lim_{c\to 2^+} \left[\frac{3}{2}(x-2)^{\frac{2}{3}}\right]_c^{10}$$

$$= \left[\frac{3}{2}(2-2)^{\frac{2}{3}} - \frac{3}{2}(1-2)^{\frac{2}{3}}\right] + \left[\frac{3}{2}(10-2)^{\frac{2}{3}} - \frac{3}{2}(2-2)^{\frac{2}{3}}\right]$$

$$= \left[0 - \frac{3}{2}\right] + \left[6 - 0\right] = \frac{9}{2}$$

<u>Example 5</u>: Determine whether the integral is proper, improper (infinite interval), improper (unbounded function).

(a) $\displaystyle\int_1^{10} \frac{1}{x+1}\,dx$ — proper

(b) $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\,dx$ — improper (infinite interval)

(c) $\displaystyle\int_1^{2} \frac{1}{\sqrt{x-1}}\,dx$ — Improper (unbounde

(d) $\displaystyle\int_0^{1} \ln x\,dx$ — Improper (unbounded

(e) $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$ — proper

(f) $\int_{2}^{4} \frac{1}{\sqrt{x^2-4}}\,dx$ — Improper (unbound.

3- Comparison Test:

If $0 \le f(x) \le g(x) \Rightarrow \int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

(ii) If $\int_a^b f(x)\,dx$ diverges, then $\int_a^b g(x)\,dx$ diverges

Example 6: Use comparison test to determine whether the integral converges or diverges.

(a) $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

Note $0 \le \sin^2 x \le 1 \rightarrow \frac{\sin^2 x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \rightarrow \int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx \le \int_0^\pi \frac{1}{\sqrt{x}}\,dx$

Now

$$\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^\pi x^{-\frac{1}{2}}\,dx = \lim_{c\to 0^+} \left[2x^{\frac{1}{2}}\right]_c^\pi = \left[2\sqrt{\pi} - 2\sqrt{c}\right] = 2\sqrt{\pi}$$

So $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx \le 2\sqrt{\pi}$, hence it converges.

(b) $\int_1^\infty \frac{x}{x^3+1}\,dx$

$$\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2} \Rightarrow \int_1^\infty \frac{x}{x^3+1}\,dx \leq \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{Converges with } p=2}$$

So $\int_1^\infty \frac{x}{x^3+1}\,dx$ Converges

(c) $\int_1^\infty \frac{x}{x^3-1}\,dx$

$\frac{x}{x^3-1} \leq \frac{1}{x^2}$, So No clue ~~how~~ to proceed using the comparsion test

wrong!

<u>Exercise 2</u>: Find $\int_{-3}^\infty \frac{1}{x^2+4x+3}\,dx$

<u>Homework</u>:

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}}$$