University of Bahrain
Department of Mathematics
MATHS312: Abstract Algebra II
Spring 2018
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Homework 1: Rings
Due on February 22, 2018
Hand in all problems

Name: _____

1. On the set of integers $\mathbb{Z}$, define the operations

$$a \oplus b := a + b - 1 \text{ and } a \odot b := a + b - ab$$

Show that $(\mathbb{Z}, \oplus, \odot)$ is a ring. Does it have a unity? Is it commutative?

2. For a prime p, prove that the set $(\mathbb{Z}_p, +_p, \cdot_p)$ is a field. (Show only that every nonzero element has a multiplicative inverse)

3. On the set of natural numbers $\mathbb{N}$, define the operations

$$a \oplus b := \max(a, b) \text{ and } a \otimes b := a + b$$

Is $(\mathbb{N}, \oplus, \otimes)$ is ring? commutative ring with unity? field?

4. (Important) Consider the set

$$\mathbb{H} := \{a + b\mathbf{i} + c\mathbf{j} + \mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$$

together with the operation

$$-1 \cdot -1 = 1, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{j}^2 = -1, \mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}, \mathbf{ji} = -\mathbf{j}, \mathbf{ik} = -\mathbf{j}, \mathbf{kj} = -\mathbf{i}$$

1. Let $w_1 = 1 + \mathbf{i} + \mathbf{j} - \mathbf{k}$, $w_2 = 2 + \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find $w_1 + w_2$ and $w_1 w_2$.

2. Generalize your work above to find $w_1 + w_2$ and $w_1 w_2$, where $w_1 = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $w_2 = a' + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$. Is the set $\mathbb{H}$ closed under the addition and multiplication?

3. Prove that $(\mathbb{H}, +, \cdot)$ is a division ring, but not a field. The set $\mathbb{H}$ is called the hamiltonian or quarterion ring.