University of Bahrain
Department of Mathematics
MATHS312: Abstract Algebra II
Spring 2018
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Homework 5: Ideals
Due on March 29, 2018
Hand in problems 1–7

Name: ____

1. Let $R = \mathbb{Z}[\sqrt{5}]$. Is the set $I = \{a + b\sqrt{5} \mid a - b \text{ is even } \}$ an ideal of R?

2. (a) Let x be an element of a commutative ring R. Show that the principal ideal generated by x is the smallest ideal of R that contains x.

(b) Conclude that the ideal generated by x is the intersection of all ideals that contain x.

(c) Let $x_1, x_2, \ldots, x_n$ be elements of a commutative ring R. Show that the ideal generated by $x_1, x_2, \ldots, x_n$ is the smallest ideal contain $x_1, x_2, \ldots, x_n$.

(d) Conclude that the ideal generated by $x_1, x_2, \ldots, x_n$ is the intersection of all ideals that contain $x_1, x_2, \ldots, x_n$.

Note: In the noncommutative ring, the ideal generated by x is defined to be the intersection of all ideals that contain x.

3. (Refer back to Homework 2, Question 4) Consider the ring of integers localized at p, i.e., $R = \mathbb{Z}_{(p)}$. Consider the set

$$p\mathbb{Z}_{(p)} := \{\frac{ap}{b} \mid \gcd(p, b) = 1\}$$

Show that $p\mathbb{Z}$ is an ideal of $\mathbb{Z}_{(p)}$.

4. (Important) Let R be a commutative ring with unity. An element a in R is called **nilpotent** if $a^n = 0$, for some positive integer n. Show that the set of all nilpotent elements of R is an ideal of R.

(b) If a is a nilpotent element, then $1 - a$ is a unit.

(Hint: Factor $1 - a^n$)

5. Let p be a prime number. Let R be a commutative ring with unity and consider the set

$$I_p := \{r \in R \mid \text{ the additive order of } r \text{ is a power of } p\}$$

Show that I_p is an ideal.

6. Let R be a commutative ring with unity and a be an element of R. Prove that the annihilator of a is in fact an ideal.