Section 1.1
System of Linear Equations

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MATHS 211: Linear Algebra

Goal:

1. To represent system of linear equations by several ways.
2. To solve system of linear equations using Jordan Gaussian Elimination.
3. To solve system of linear equations using the inverse of a matrix.

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1- Representing Linear System as augmented matrix and matrix form

Example 1

Represent the linear system in two forms

$$2x - 7y = -1$$
$$x + 3y = 6$$

2- Solving System of Linear Equations using elementary row operations

Example 2

Solve the system

$$2x - 7y = -1$$
$$x + 3y = 6$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{cc|c} 2 & -7 & -1 \\ 1 & 3 & 6 \end{array}\right), \qquad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & -7 & -1 \end{array}\right), \qquad R_2 \rightarrow R_2 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2-2(1) & -7-2(3) & -1-2(6) \end{array}\right)$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 2-2(1) & -7-2(3) & | & -1-2(6) \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 0 & -13 & | & -13 \end{pmatrix} \qquad R_2 \to \frac{1}{-13} R_2$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 0 & 1 & | & 1 \end{pmatrix} R_1 \to R_1 - 3R_2$$

$$\begin{pmatrix} 1-3(0) & 3-3(1) & | & 6-3(1) \\ 0 & 1 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 1 \end{pmatrix}$$

So $x = 3$ and $y = 1$ and thus the solution set is $\{(3, 1)\}$

Example 3

Solve the system

$$\begin{aligned} x + 4y &= 9 \\ 3x - y &= 6 \\ 2x - 2y &= 4 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 2 & -2 & 4 \end{array}\right), \qquad R_2 \to R_2 - 3R_1 \quad R_3 \to R_3 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3-3(1) & -1-3(4) & 6-3(9) \\ 2-2(1) & -2-2(4) & 4-2(9) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -10 & -14 \end{array}\right) \qquad R_2 \to \frac{1}{-13} R_2$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & -10 & | & -14 \end{pmatrix}, \qquad R_3 \to R_3 + 10R_2$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & -10+10(1) & | & -14+10(\frac{-21}{-13}) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & 0 & | & \frac{28}{13} \end{pmatrix}$$

We have $0 = \frac{28}{13}$ which is a false statement and thus there will be no solution.

Example 4

Solve the system

$$\begin{aligned} x + y - z &= 7 \\ 4x + 6y - 4z &= 8 \\ x - y - 5z &= 23 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4 & 6 & -4 & 8 \\ 1 & -1 & -5 & 23 \end{array}\right), \quad R_2 \to R_2 - 4R_1 \quad R_3 \to R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4-4(1) & 6-4(1) & -4-4(-1) & 8-4(7) \\ 1-1 & -1-1 & -5-(-1) & 23-7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & 0 & -20 \\ 0 & -2 & -4 & 16 \end{array}\right) \qquad R_2 \to \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & -2 & -4 & 16 \end{array}\right), \qquad R_3 \to R_3 + 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & -2+2(1) & -4+2(0) & 16+2(-10) \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & -4 & -4 \end{array}\right) R_3 \to \frac{1}{-4} R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad R_1 \to R_1 + R_3$$

$$\left(\begin{array}{ccc|c} 1+1(0) & 1+1(0) & -1+1(1) & 7+1(1) \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & -10 \end{array}\right. \qquad R_1 \to R_1 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad R_1 \to R_1 - R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

So $x = 18$, $y = -10$, and $z = 1$.
Solution Set $= \{(18, -10, 1)\}$.

Example 5

Solve the system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 7y &= 4 \\ 3x + 15y + 3z &= 15 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 2 & 7 & 0 & 4 \\ 3 & 15 & 3 & 15 \end{array}\right), \qquad R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 3R_1$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 2-2(1) & 7-2(3) & 0-2(0) & 4-2(2) \\ 3-3(1) & 15-3(3) & 3-3(0) & 15-3(2) \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 3 & 9 \end{array}\right) \qquad R_3 \to R_3 - 6R_2$$

3 - Solving Linear System using the inverse of a matrix

Example 6

Solve

$$\begin{aligned} 3x + y &= 2 \\ 4x + y &= 3 \end{aligned}$$

Solution: This system can be written in a matrix multiplication form as

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$I_2 \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise 7

Solve the following system using the inverse matrix method.

$$2x - 3y = 9$$
$$4x + y = 1$$