Section 2.1
Determinant of a matrix

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MATHS 211: Linear Algebra

Goal:

1. To define the determinant of a matrix.
2. To find the determinant of a matrix using cofactor expansion (Section 2.1).
3. To find the determinant of a matrix using row reduction (Section 2.2).
4. Explore the properties of the determinant and its relation to the inverse. (Section 2.3)
5. To solve linear system using the Cramer's rule. (Section 2.3)

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1 - Definition of the determinant of a matrix

Example 1

(Motivational Example) Find the inverse of the following 2×2 matrix and state the condition that it will be exist.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Determinant of 2×2 matrix

Definition 2

The number $ad - bc$ is called the *determinant* of A

Example 3

Let

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

Find all minors and cofactors.

Example 4

Find $\det(A)$ for

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 3 & 5 \\ 7 & 0 & 6 \end{pmatrix}$$

Solution:

General Definition of the determinant

Definition 5

The determinant of a matrix is a *signed* sum of the product of exactly one element from each row provided that we do not use the same column twice.

Definition 6

Let A be a square matrix.

1. **The minor of entry** a_{ij} is denote by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A.
2. **The cofactor entry of entry** a_{ij} is denoted by C_{ij} and equals $(-1)^{i+j}M_{ij}$.

Note that $C_{ij} = \pm M_{ij}$ depending on the "checkerboard":

$$\begin{pmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ \dots & \dots & \dots & \dots & & \end{pmatrix}$$

Definition 7

Let A be a square matrix, **determinant of** A, denoted by $\det(A)$ is sum of the entries in any row or column multiplied by their cofactors, i.e.,

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{colum–wise}$$

or

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{row–wise}$$

Example 8

Find $\det(A)$ for

$$A = \begin{pmatrix} 5 & 2 & -2 & 0 \\ 3 & 2 & -2 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 5 & 7 \end{pmatrix}$$

Solution:

Example 9

Find the determinant for following matrices

$$A = \begin{pmatrix} 2 & -12 & 14 \\ 0 & 3 & 50 \\ 0 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} 5 & 0 & 0 \\ -3 & -3 & 0 \\ 7 & 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Solution:

Theorem 10

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ *is the product of the entries on the main diagonal of the matrix, that is* $\det(A) = a_{11}a_{22}\dots a_{nn}$.