

Section 4.1

Vector Spaces

Dr. Abdulla Eid

College of Science

MATHS 211: Linear Algebra

Goal:

- 1 Define Vector spaces.
- 2 Explore important examples of vector spaces

Dr. Abdulla Eid

Vectors

- In short, vectors are elements that we can **add** and **multiply by a scalar**.
- Let V be any arbitrary nonempty set of elements together with two operations $(+, \cdot)$ (**addition** and **multiplication by a scalar**).

$$\begin{aligned} + : V \times V &\rightarrow V \\ (\mathbf{u}, \mathbf{v}) &\mapsto \mathbf{u} + \mathbf{v} \end{aligned}$$

and

$$\begin{aligned} \cdot : \mathbb{R} \times V &\rightarrow V \\ (k, \mathbf{u}) &\mapsto k \cdot \mathbf{u} \end{aligned}$$

We say $(V, +, \cdot, \mathbb{R})$ a real vector space if it satisfies the following axioms

Axioms

- 1 (Closure) If \mathbf{u}, \mathbf{v} in V , then so $\mathbf{u} + \mathbf{v}$ in V .
- 2 (commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- 3 (Associative) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- 4 (Zero) There is an element $\mathbf{0}$ in V called the **zero vector** such that $\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u}$.
- 5 (Negative) For each \mathbf{u} , there is an element $(-\mathbf{u})$ in V called the **negative vector** of \mathbf{u} such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0} = (-\mathbf{u}) + \mathbf{u}$.
- 6 If $k \in \mathbb{R}$ and $\mathbf{u} \in V$, then so $k \cdot \mathbf{u} \in V$.
- 7 $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.
- 8 $(k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$.
- 9 $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$.
- 10 $1\mathbf{u} = \mathbf{u}$.

Premier Example, \mathbb{R}^n

Example 1

Consider the set

$$\mathbb{R}^n := \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

$$k\mathbf{u} = k(a_1, a_2, \dots, a_n) = (ka_1, ka_2, \dots, ka_n)$$

Dr. Abdulla Eid

Identification

We usually identify \mathbb{R}^n with the set of all columns vector of length n , i.e.,

$$\mathbb{R}^n = \text{Mat}(n, 1, \mathbb{R}) \cong \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$$

Of course, we can identify \mathbb{R} with the set of all row vectors of length n .

Dr. Abdulla Eid

Zero Vector Space

Example 2

Consider the set

$$V := \{*\}$$

$$\mathbf{u} + \mathbf{v} = * + * = *$$

$$k\mathbf{u} = k \cdot * = *$$

This is called the *zero vector space* and usually we denote it by $V = \{\mathbf{0}\}$.

Premier Example, $\text{Mat}(m, n, \mathbb{R})$

Example 3

Consider the set

$$\text{Mat}(m, n, \mathbb{R}) := \{(a_{ij}) \mid a_{ij} \in \mathbb{R}, i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$$

$$\mathbf{u} + \mathbf{v} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$k\mathbf{u} = k(a_{ij}) = (ka_{ij})$$

Premier example, \mathbb{P}_n

Example 4

Consider the set

$$\mathbb{P}_n := \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in \mathbb{R}\}$$

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \end{aligned}$$

$$k\mathbf{u} = k(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$

Dr. Abdulla Eid

Example, \mathbb{R}^∞

Example 5

Consider the set

$$\mathbb{R}^\infty := \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots)$$

$$k\mathbf{u} = k(a_1, a_2, \dots) = (ka_1, ka_2, \dots)$$

The vector space of infinite sequences of real numbers.

Premier Example, Maps(\mathbb{R}, \mathbb{R})

Example 6

Consider the set

$$\text{Maps}(\mathbb{R}, \mathbb{R}) := \{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (f + g)(x) = f(x) + g(x)$$

$$k\mathbf{u} = (kf)(x) = kf(x)$$

The vector space of all real valued function on \mathbb{R} .

Unusual Example, \mathbb{R} with nonstandard operations

Example 7

Consider the set

$$\mathbb{R} := \{a \mid a \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = a + b = ab$$

$$k\mathbf{u} = ka = a^k$$

This defined indeed a vector space of real numbers. **Why?**

Dr. Abdulla Eid

Non-Example, \mathbb{R}^2 with nonstandard operations

Example 8

Consider the set

$$\mathbb{R}^2 := \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$k\mathbf{u} = k(a_1, a_2) = (0, ka_2)$$

This is **not** a vector space of infinite sequences of real numbers. **Why?**

Properties of Vector spaces

Theorem 9

Let $(V, +, \cdot, \mathbb{R})$ be a vector space, $\mathbf{u} \in V$ and $k \in \mathbb{R}$. Then

- 1 $0\mathbf{u} = \mathbf{0}$
- 2 $k\mathbf{0} = \mathbf{0}$
- 3 $(-1)\mathbf{u} = (-\mathbf{u})$
- 4 if $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$