Section 4.5
Dimension

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MATHS 211: Linear Algebra

Goal:

1. Define the dimension of a vector space.

Basis

Definition 1

Let V be a vector space. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a **basis**. The number n is called the **dimension** of the vector space V, and denoted by $\dim(V)$.

Note: This number n is independent of the chosen basis

Theorem 2

All bases for a finite–dimensional vector space have the same number of vectors.

Lemma 3

Let V be a vector space. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis.

1. *If a set has more than n vectors, then it is linearly dependent.*
2. *If a set has fewer than n vectors, then it does not span V.*

Standard Basis

Note:

- The standard basis for $\mathbb{R}^2$ is e_1, e_2, where
$$e_1 = (1,0) \text{ and } e_2 = (0,1)$$
So $\dim(\mathbb{R}^2) = 2$.
- The standard basis for $\mathbb{R}^3$ is e_1, e_2, e_3, where
$$e_1 = (1,0,0),\ e_2 = (0,1,0) \text{ and } e_3 = (0,0,1)$$
$\dim(\mathbb{R}^3) = 3$.
- The standard basis for $\mathbb{R}^n$ is $e_1, e_2, e_3, \ldots, e_n$, where
$$e_1 = (1,0,\ldots,0),\ e_2 = (0,1,\ldots,0),\ e_3 = (0,0,1,0,\ldots,0) \text{ and }$$
$$e_n = (0,0,\ldots,1)$$
$\dim(\mathbb{R}^n) = n$.
- The standard basis for $\mathbb{P}_2$ is $1, X, X^2$.
$\dim(\mathbb{P}_2) = 3$.
- The standard basis for $\mathbb{P}_n$ is $1, X, X^2, \ldots, X^n$.
$\dim(\mathbb{P}_n) = n + 1$.

The standard basis for $\text{Mat}(2, 2, \mathbb{R})$ is
$\dim(\text{Mat}(2, 2, \mathbb{R})) = 4$. and in general $\dim(\text{Mat}(m, n, \mathbb{R})) = mn$.

Example 4

Find bases for the subspace of $\mathbb{R}^3$ given by the plane $2x + 4y - 3z = 0$

Example 5

Find bases for the subspace of $\mathbb{R}^3$ given by the plane $x + z = 0$

Example 6

Find bases for the subspace of $\mathbb{R}^3$ given by the plane
$x = 4t, y = 2t, z = -t$.

Example 7

Find bases for the subspace of $\mathbb{R}^3$ given by all vectors of the form (a, b, c), where $c = a - b$.

Example 8

Find the dimension of the subspace W of $\mathbb{R}^4$, given by all vectors of the form $(0, a, b, c)$.

Example 9

Find the dimension of the subspace W of $\mathbb{R}^4$, given by all vectors of the form (a, b, c, d), where $d = a + 2b$, $c = 3a - b$.

Example 10

Find the dimension of the subspace W of $\mathbb{R}^4$, given by all vectors of the form (a, b, c, d), where $c = a$, $b = d = -a$.

Example 11

Find the dimension of the subspace W of $\mathrm{Mat}(n, n, \mathbb{R})$, given by all diagonal matrices.

Example 12

Find the dimension of the subspace W of $\mathrm{Mat}(n, n, \mathbb{R})$, given by all symmetric matrices.

Example 13

Find the dimension of the subspace W of $\mathrm{Mat}(n, n, \mathbb{R})$, given by all upper triangular matrices.

Example 14

Find the dimension of the subspace W of $\mathbb{P}_n$, given by all polynomials with a horizontal tangent at $x = 0$.

Infinite Dimensional Spaces

Example 15

Let $V = \text{Maps}(\mathbb{R}, \mathbb{R})$. Show that for every positive integer n, one can find $n+1$ independent functions.

Conclusion: $\text{Maps}(\mathbb{R}, \mathbb{R})$ is infinite–dimensional. So do $C^i(\mathbb{R}, \mathbb{R})$ and $C^\infty(\mathbb{R}, \mathbb{R})$.

Plus/Minus Theorem

Theorem 16

Let S be a nonempty set of vectors in a vector space V.

1. *If S is a linearly independent set, and $\mathbf{v} in V$ that is outside $Span(S)$, then the set $S \cup \{\mathbf{v}\}$ is still linearly independent.*
2. *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, then*

$$Span(S) = Span(S - \{\mathbf{v}\})$$

Example 17

Enlarge the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\}$ to produce a basis for $\mathbb{R}^3$.

Example 18

Enlarge the set $\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ to produce a basis for $\mathbb{R}^3$.