Section 6.1
Inner Product

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MATHS 211: Linear Algebra

Goal:

1. Define Inner Product.
2. Examples of the inner product.
3. Properties of the inner product.

Definition 1

General Inner Product An **inner product** is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. (Symmetric axiom)
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$. (Additive axiom)
3. $\langle k\mathbf{u}, \mathbf{w} \rangle = k \langle \mathbf{u}, \mathbf{w} \rangle$. (homogeneity axiom)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle (, \mathbf{u} \rangle , \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$ (positive definiteness axiom)

$\mathbb{R}^2$

Example 2

Let $V = \mathbb{R}^2$. Define
$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$$

Example 3

Find $\left\langle \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix} \right\rangle$,
$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$,
$\left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$

$\mathbb{R}^n$

Example 4

Let $V = \mathbb{R}^2$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is called the **dot** or **Euclidean** product.

Example 5

Find $\left\langle \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 11 \\ -2 \\ 5 \end{pmatrix} \right\rangle$,

$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle$,

$\left\langle \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} -2 \right\rangle$

$\mathbb{R}^2$

Example 6

Let $V = \mathbb{R}^2$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$$

Example 7

Find $\left\langle \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix} \right\rangle$,
$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$,
$\left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$

Weighted inner product $\mathbb{R}^n$

Example 8

Let $V = \mathbb{R}^n$ and let $w_1, \dots, w_n$ be positive real numbers (**weights**). Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

This is called the **dot** or **Euclidean** product.

Norm and distance

Definition 9

Let V be a real inner space. The **norm** (or **length**) of a vector $\mathbf{v}$ of V is denoted by $||\mathbf{v}||$ and is defined by

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the distance between any two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a **unit vector**.

$\mathbb{R}^2$

Example 10

Let $V = \mathbb{R}^2$ with the Euclidean product. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and $k = 5$. Find the following:

(a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle k\mathbf{u}, \mathbf{v} \rangle$
(c) $\langle u + v, w \rangle$ (d) $||\mathbf{u}||$
(e) $d(\mathbf{u}, \mathbf{v})$ (f) $||\mathbf{u} - k\mathbf{v}||$

$\mathbb{R}^2$

Example 11

Let $V = \mathbb{R}^2$.Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$$

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and $k = 5$. Find the following:

(a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle k\mathbf{u}, \mathbf{v} \rangle$

(c) $\langle u + v, w \rangle$ (d) $||\mathbf{u}||$

(e) $d(\mathbf{u}, \mathbf{v})$ (f) $||\mathbf{u} - k\mathbf{v}||$

Matrix inner product on $\mathbb{R}^n$

Example 12

Let $V = \mathbb{R}^n$ and let A be invertible $n \times n$ matrix. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

This is called the **inner product on $\mathbb{R}^n$ generated by** A
The Euclidean inner product is a special case with $A = I_n$ and the weighted inner product is a special case with

$$A = \begin{pmatrix} \sqrt{w_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{w_2} & 0 & \dots & 0 \\ \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \sqrt{w_n} \end{pmatrix}$$

$\mathbb{R}^2$

Example 13

Let $V = \mathbb{R}^2$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Find the following:

(a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle \mathbf{u}, \mathbf{v} \rangle$

(c) $||u||$ (d) $||\mathbf{u}||$

(e) $d(\mathbf{u}, \mathbf{v})$ (f) $||\mathbf{u} - \mathbf{v}||^2$

Inner product on square matrices

Definition 14

Let $V = \text{Mat}(m, n, \mathbb{R})nn$ and let U, V be invertible $n \times n$ matrices. Define

$$\langle U, V \rangle = tr\left(U^T V\right)$$

if U, V are two by two matrices, then what is $\langle U, V \rangle$?

Inner product on matrices

Example 15

Find $< U, V >$ for

1. $U = \begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}, V = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$
2. $U = \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}, V = \begin{pmatrix} 4 & 6 \\ 0 & 8 \end{pmatrix}$

Inner product on Polynomials

Definition 16

Let $V = \mathbb{P}_n$. Let $p = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$ and $q = b_0 + b_1X + b_2X^2 + \cdots + b_nX^n$ be two polynomials. Define the **standard inner product** to be

$$\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

If $x_0, x_1, \ldots, x_n$ are distinct real numbers, define the **evaluation inner product** to be

$$\langle p, q\rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + \cdots + p(x_n)q(x_n)$$

Inner product on matrices

Example 17

Find $< p, q >$, $||p||$, $||q||^2$, and $d(p, q)$ for

1. $p = 3 - X + 2X^2, q = 2 - 4X^2$.
2. $p = X + 3X^2, q = 2 - X + 4X^2$.
3. $o = 1 - 2X + 3X^2, q = 4 + X^2$ with $x_0 = 2, x_1 = -1, x_2 = 1$.

Inner Product on $C([a, b])$

Example 18

Let $V = C([a, b])$ be the vector space of continuous functions on an interval $[a, b]$. Define

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

Let $f(x) = x + 1$, $g(x) = x - 1$, $h(x) = 5$, and $a = 1, b = 2$. Find the following:

(a) $\langle \mathbf{u}, \mathbf{w}\rangle$ (b) $\langle 8\mathbf{u}, \mathbf{v}\rangle$

(c) $\langle u + v, w\rangle$ (d) $||\mathbf{u}||$

Non–example, Lorentian Inner product

Example 19

Let $V = \mathbb{R}^4$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2 - t_1 t_2$$

This is called the **Lorentzian** inner product. This is of central importance in Einstein's theory of special relativity.
Note: This is not an inner product! Why?

Example 20

Find $\left\langle \begin{pmatrix} -5 \\ 3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 11 \\ -2 \\ 5 \\ 2 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 2 \end{pmatrix} \right\rangle$.

Continue

Two space time vectors $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}$ are

Seperated by a distance $\sqrt{\langle X_1, X_2 \rangle}$ if $\langle X_1, X_2 \rangle \geq 0$

Seperated by a time $\sqrt{-\langle X_1, X_2 \rangle}$ if $\langle X_1, X_2 \rangle \leq 0$

Theorem 21

Norm and Distance Let V be an inner product vector space. Then,

1. $\langle \mathbf{0}, \mathbf{v} \rangle = \mathbf{0}$.
2. $||\mathbf{v}|| \geq 0$ *with equality if and only if* $\mathbf{v} = \mathbf{0}$.
3. $||k\mathbf{v}|| = |k|||\mathbf{v}||$.
4. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

Example 22

Prove that in any inner product vector space, we have

$$||\mathbf{u}+\mathbf{v}||^2+||\mathbf{u}-\mathbf{v}||^2=2||u||^2+2||\mathbf{v}||^2$$

Example 23

Prove that in any inner product vector space, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}-\mathbf{v}||^2 = 2||u||^2 + 2||\mathbf{v}||^2$$