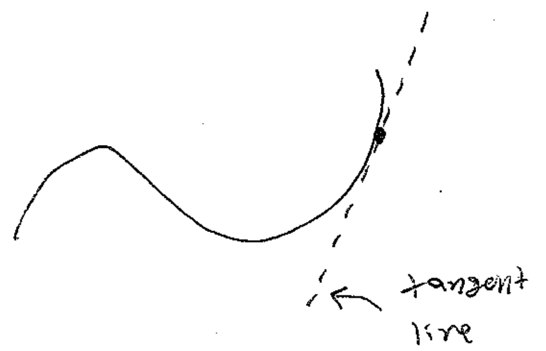
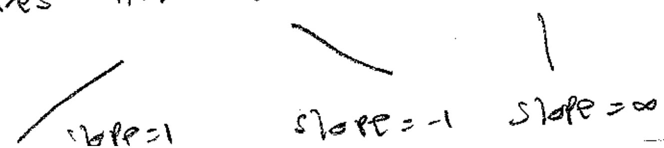
Calculus

Derivative

- we find the <u>slope</u> of the <u>tangent</u> <u>line</u> of a function at certain point.

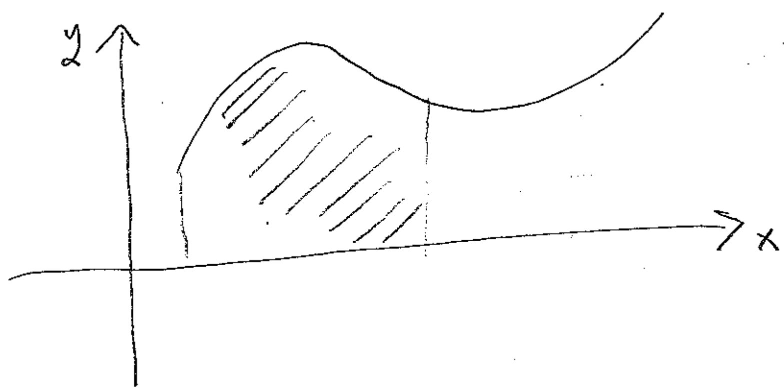


- <u>Slope</u> of a line is a number that measures how the line looks like, e.g.,



Integration

- we find the <u>area</u> under the graph of a function



Question : What is the relation between differentiati and integration? In other words, what is the relation between finding the slope of a tangent line (derivative) and finding the area under the graph (Integration)

Answer : Surprisingly enough, these two operation are reversing each other. So if we integrate, then we differentiate we get the original function back and if we differentiate first and integrate, we get again the original function back!
This is what the "Fundamental Theorem of Calculus" says!

$f(x)$ --- original function

differentiate (finding the slope) ↓

$f'(x)$ --- function of the slopes

Integrate (finding area) ↓

$f(x)$ --- original function

$f(x)$ --- original

Integrate (finding area) ↓

$F(x)$ --- function of the areas

differentiate (Finding the slope) ↓

$f(x)$ --- original function

The common ingrident in defining derivative & Integral is the concept of the limit

§2.2 - limit of a function and limit laws

① Definition of ~~the~~ limit

Motivation Example:

How does the function $f(x) = \dfrac{x^2-1}{x-1}$ behave at $x=1$?

, $f(1) = \dfrac{1^2-1}{1-1} = \dfrac{0}{0} \leftarrow$ undefined! so we can't substitute directly with $x=1$, so instead we ~~get~~ closer to 1 and check the values!

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.99	—	2.001	2.01	2.1

2

So $f(x)$ approaches 2 as x approaches 1.

we write that $$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Question: In order to find the limit of a function, do we need to make table like the above one?

Answer: NO, you will learn algebraic method to find the limit. But for now try to find the limit of these functions using the table and write the general rule.

Exercise 1: Find the limit of the following:

(a) $\lim_{x \to 5} x$ (b) $\lim_{x \to x_0} x$

(c) $\lim_{x \to 5} 7$ (d) $\lim_{x \to x_0} K$, K is constant

Question: Does the limit always exist? No!

Example 2: Find the limit of the following functions as $x \to 0$

(a) $$U(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x \geq 0 \end{cases}$$

(b) $$g(x) = \begin{cases} \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

(c) $$f(x) = \begin{cases} 0 & , x \leq 0 \\ \sin \frac{1}{x} & , x > 0 \end{cases}$$

Solution:

(a)

x	-0.1	-0.01	0	0.01	0.1
f(x)	0	0	1	1	1

0 1

Limit has two options either 0 or 1, so the limit itself doesn't exist.

② The limit Rules

Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ (Both exist)

1. Sum Rule :- $\lim_{x\to c}\left[f(x) \pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

$$= L + M.$$

2. Product Rule :- $\lim_{x\to c}\left[f(x)\cdot g(x)\right] = \lim_{x\to 0} f(x) \cdot \lim_{x\to 0} g(x) = M \cdot L$

3. Quotient Rule :- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$.

4. Power Rule :- $\lim_{x\to c} \left(f(x)\right)^n = \left(\lim_{x\to c} f(x)\right)^n = L^n$

5. Root Rule :- $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$.

Example 3 :-

(a) $\lim_{x\to 3} x^3 = \left(\lim_{x\to 3} x\right)^3 = (3)^3 = 27$

(b) $\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$

(c) $\lim_{x\to .2} \sqrt{4x^2 - 3} = \sqrt{4c^2 - 3}$

To find $\lim\limits_{x \to c} f(x)$

1- Direct substitute c to check if there are No problems.

2- Use algebraic method to clear any problems.

Example 4: Find the limit (Com...

$\boxed{1}$ $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{x^2-x} = \dfrac{0}{0}$ undefined

$= \lim\limits_{x \to 1} \dfrac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} = \lim\limits_{x \to 1} \dfrac{(x+2)}{x} = \dfrac{3}{1}$

$= 3$.

(A) eliminating zero denominator by cancelling common factor

$\boxed{4}$ $\lim\limits_{x \to 5} \dfrac{x-5}{x^2-25}$

$\boxed{3}$ $\lim\limits_{y \to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$

$\boxed{2}$ $\lim\limits_{x \to 3} \dfrac{x^2-3x}{x^2-9}$

6 $\lim\limits_{x \to 9} \dfrac{\sqrt{x}-3}{x-9}$

$\boxed{5}$ $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{3x^2-x-2}$

(B) Conjugate and multiplying by 1.

Example 5:

$\boxed{7}$ $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+100}-10}{x^2} = \dfrac{0}{0}$ undefined!

$\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+100}-10}{x^2} \cdot \underbrace{\dfrac{(\sqrt{x^2+100}+10)}{(\sqrt{x^2+100}+10)}}_{=1} = \lim\limits_{x \to 0} \dfrac{\cancel{x^2}+100-100}{\cancel{x^2}(\sqrt{x^2+100}+10)} = \dfrac{1}{20} = 0.05$

Exercise Find the limit of each of the following:

(a) $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8} - 3}{(x+1)}$ (b) $\lim\limits_{x \to 4} \dfrac{4-x}{(5-\sqrt{x^2+9})}$.

The Sandwich Theorem :- (Squeeze Theorem)

Suppose $g(x) \leq \underbrace{f(x)}_{} \leq h(x)$. Suppose

f(x) is squeezed between g(x) & h(x).

$\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$. Then $\lim\limits_{x \to c} f(x) = L$.

Examples :

(a) Find $\lim\limits_{x \to 0} u(x)$ if $1 - \dfrac{x^2}{4} \leq u(x) \leq 1 + \dfrac{x^2}{2}$

we take the limit as $x \to 0$ to get :

$$\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) \leq \lim_{x \to 0} u(x) \leq \lim_{x \to 0} \left(1 + \frac{x^2}{2}\right)$$

$$1 \leq \lim_{x \to 0} u(x) \leq 1 \rightsquigarrow \lim_{x \to 0} u(x) = 1$$

Exercise :

(a) $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$. Find $\lim\limits_{x \to 0} f(x)$?

(b) $\lim\limits_{x \to 0} x \sin x$?