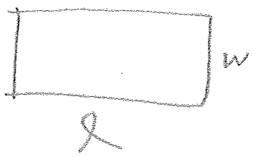
§ 4.6 - Optimization

Example 1: What is the smallest perimeter possible for a rectangle whose area is 4 cm^2? What are the dimensions?

Solution: Given: area $= 4 \to lw = 4 \to l = \frac{4}{w}$



Required: minimize perimeter

minimize $P = 2l + 2w$

$$P = 2 \cdot \frac{4}{w} + 2w = \frac{8}{w} + 2w.$$

We need to find the local minimum, so $P'(w) = -\frac{8}{w^2} + 2$

$P' = 0$ or P' DNE

$-\frac{8}{w^2} + 2 = 0$

$W = 0$. rejected!

$2 = \frac{8}{w^2}$

$2w^2 = 8$

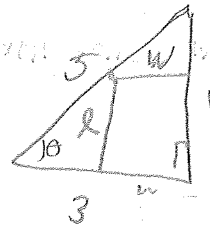
$w^2 = 4$

$\boxed{w = 2}$, $w = -2$ rejected!

Now we check $w = 2$ is local minimum by the second derivative test, so $P''(2) = \frac{-16}{(2)^3} < 0$, so $w = 2$, $l = 2$ and $\boxed{P = 8 \text{ cm}}$ 1

Exercise 1: Find the smallest area of a rectangle whose perimeter is 32 cm.

Example 2: Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the figure.



Solution:

Given: $\tan\theta = \frac{4}{3} = \frac{l}{3-w} \rightarrow 4(3-w) = 3l \Rightarrow l = \frac{4}{3}(3-w)$.

Required: Minimize: Area $= lw$

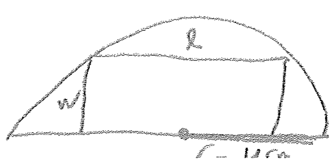
$$A(w) = \frac{4}{3}(3-w)w = 4w - \frac{4}{3}w^2$$

$A'(w) = 0 \rightarrow 4 - \frac{8}{3}w = 0 \rightarrow w = \frac{12}{8} = \frac{3}{2}$.

$A''(w) = -\frac{8}{3} < 0$, so we here local maximum.

$w = \frac{3}{2}$ cm, $l = 2$ cm and Area $= 3$ cm^2

Exercise 2: same as above, but



Example 3: Find a positive number for which the Sum of its ~~and~~ its reciprocal is the smallest possible.

Solution:

Required: minimize $S(x) = x + \frac{1}{x}$

$$S'(x) = 1 - \frac{1}{x^2}$$

$S'(x) = 0$ $\qquad\qquad$ $S'(x)$ DNE if $x = 0$ (rejected)

$$1 - \frac{1}{x^2} = 0 \rightarrow x^2 = 1$$

$$\boxed{x = 1} \text{ or } \boxed{x = -1}$$

To find which one is a local minimum, we use the 2^{nd} derivative test to get

$$S''(x) = +\frac{2}{x^3}, \text{ so } \quad S''(1) = +2 > 0 \text{ --- maximum}$$

$$S''(-1) = -2 < 0 \text{ --- minimum.}$$

Hence $x = +1$ is a local minima.

Exercise 3: Find a positive number for which the sum and its reciprocal and four times its square is the smallest possible.

Example 4: For what values of a and b make

$$f(x) = x^3 + ax^2 + bx$$ have

(a) local maximum at $x=-1$ and local minimum at $x=3$

Solution:

$$f'(x) = 3x^2 + 2ax + b$$

$$f''(x) = 6x + 2a$$

Now $f'(-1)=0$ and $f'(3)=0$

$3 - 2a + b = 0$ $\qquad$ $27 + 6a + b = 0$

$\boxed{a = 3}$

$\boxed{b = 3}$.

(b) (exercise) a local minimum at $x=4$ and a point of inflection at $x = 1$?