University of Bahrain
Department of Mathematics
MATHS253: Set Theory
Fall 2018
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Homework 14: Relations
Due December 27, 2018

Name: _____

1. Write the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where aRb if
    1. $a = b$.

    2. $a + b = 4$.

    3. $a < b$.

4. a divides b.

2. For each of the following relations on the set $\{1,2,3,4\}$, decide whether it is reflexive, symmetric, and transitive. (Write only the final answer here and attach a separate paper with the solution

   1. $R_1 = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$..
   2. $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$.
   3. $R_3 = \{(2,4),(4,2)\}$.
   4. $R_4 = \{(1,2),(2,3),(3,4)\}$.
   5. $R_5 = \{(1,1),(2,2),(3,3),(4,4)\}$.
   6. $R_6 = \{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$.

3. Determine whether the relation R on the set of all people is reflexive, symmetric, and transitive.

    1. $aRb :\iff a$ is taller than b.
    2. $aRb :\iff a$ and b were born on the same day.
    3. $aRb :\iff a$ has the same first name as b.
    4. $aRb :\iff a$ and b have a common grandparent.

4. Let R, S be two relations on A. Prove the following:

   (a) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

   (b) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

   (c) $R \cup R^{-1}$ is symmetric.

5. Let R be a relation defined on $\{1,2,3,4\}$ by

$$R = \{(1,3),(1,4),(2,2),(2,4),(3,1),(3,2),(4,4)\}$$

   (a) Compute R^{-1}.

   (b) Compute the relations $R \cup R^{-1}$ and $R \cap R^{-1}$ and check that they are symmetric.

6. For the relation $R = \{(x,y) : x \leq y\}$ defined on $\mathbb{N}$, what is R^{-1}?

7. (Composition of relations) Suppose that R, S are relations on some set A. Define the *composition* of R and S to be the new relation $R \circ S$ defined by

$$(a, c) \in R \circ S : \iff \exists b \in A, \quad aRb \wedge bSc$$

(a) If $R = \{(1,1), (1,2), (2,3), (3,1), (3,3)\}$ and $S = \{(1,2), (1,3), (2,1), (3,3)\}$, find $R \circ S$.

(b) Suppose that R and S are reflexive. Prove that $R \circ S$ is reflexive.

(c) Give an example of two symmetric relations R and S such that $R \circ S$ is *not* symmetric.

(d) Suppose that R and S are symmetric. Prove that $(a, c) \in R \circ S$ if and only if $(c, a) \in S \circ R$.

8. (Requires Linear Algebra) Let $A = \{1, 2, 3\}$ and define two relations R_1 and R_2 by

$$R_1 = \{(1,2),(2,1),(2,2),(2,3),(3,1)\}, \quad R_2 = \{(1,2),(2,2),(2,3),(3,1),(3,2),(3,2)\}$$

(a) Find the matrices M_{R_1} and M_{R_2} associated to the relations R_1 and R_2.

(b) Find the matrix represents $R_1 \cap R_2$ (Hint: $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$)

(c) Find the matrix represents $R_1 \cup R_2$ (Hint: $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$)

(d) Find the matrix represents $R_2 \circ R_1$ (Hint: $M_{R_2 \circ R_1} = M_{R_1} \odot M_{R_2}$, boolean matrix multiplication)

(e) Find the matrices represents $R_1^2 = R_1 \circ R_1$ and $R_1^3 = R_1 \circ R_1 \circ R_1$.