§8.1 - Integration by parts

Recall: The product rule

$$d(uv) = u\,dv + v\,du \Rightarrow u\,dv = d(uv) - v\,du$$

$$\int u\,dv = \int d(uv) - \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Note: we choose u (Easily differentiable, hard to integrat

v (Easily integrable)

Example 1: $\int x e^x\,dx$

$u = x \quad , dv = e^x\,dx$

$du = dx \quad , v = e^x$

$$\int u\,dv = uv - \int v\,du$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C.$$

Example 2: $\int x e^{-x} dx$

$u = x \quad , dv = e^{-x}$

$du = dx \quad , v = -e^{-x}$

$\int u\,dv = uv - \int v\,du$

$= -xe^{-x} - \int -e^{-x} dx$

$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + c$

Example 3: $\int x \sin x \, dx$

$u = x \quad , dv = \sin x \, dx$

$du = dx \quad , v = -\cos x$

$\int u\,dv = uv - \int v\,du$

$= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$

Example 4: $\int x^2 e^x dx$

$u = x^2 \quad , dv = e^x dx$

$du = 2x\,dx \quad , v = e^x$

$\int u\,dv = uv - \int v\,du$

$= x^2 e^x - \int 2x e^x dx = x^2 e^x - 2 \int x e^x dx$ (use Exampl 1)

$x^2 e^x - 2xe^x + 2e^x + c$

Example 5: $\int \ln x \, dx$

$u = \ln x$, $dv = 1dx$

$du = \frac{1}{x} dx$, $v = x$

$\int u dv = uv - \int v du$

$= x \ln x - \int x \cdot \frac{1}{x} d\cancel{x} = x \ln x - \int 1 d\cancel{x}$

$= x \ln x - x + C$

Example 6: $\int x \ln x \, dx$

$u = \ln x$, $dv = x dx$

$du = \frac{1}{x} dx$, $v = \frac{x^2}{2}$

$\int u dv = uv - \int v du$

$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

Exercise: $\int (\ln x)^2 dx$

Note: To choose u, we might follow this rule

LIPET

logarithmic, Inverse, Polynomial, Trigonometric $\longleftrightarrow$ Exponential

Example 7: $\int \tan^{-1} x \, dx$

$u = \tan^{-1} x \quad , \quad dv = dx$

$du = \frac{1}{1+x^2} dx \quad , \quad v = x$

$\int u \, dv = uv - \int v \, du$

$= x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2} dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx = x \tan^{-1} x + \frac{1}{2} \ln(1+x^2) + c$

Example 8: "Repeated use of Integration by parts"

$I = \int e^x \sin x \, dx$

$u = e^x \quad , \quad dv = \sin x \, dx$

$du = e^x dx \quad , \quad v = -\cos x$

$I = uv - \int v \, du$

$I = -e^x \cos x + \int e^x \cos x \, dx \quad \text{———} \quad (1)$

$u = e^x \quad , \quad dv = \cos x \, dx$

$du = e^x dx \quad , \quad v = \sin x$

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

So $I = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$$I = -e^x \cos x + e^x \sin x - I$$

$$2I = -e^x \cos x + e^x \sin x$$

$$\boxed{I = \frac{-e^x \cos x + e^x \sin x}{2}}$$

Exercise 2:

$$\int e^{2x} \cos x \, dx$$

Exercise 3:

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

Exercise 4: Without using $\sin^2\theta = \frac{1-\cos 2x}{2}$.

(a) Use integration by parts to reduce $\int \sin^2\theta \, d\theta$ into an integral involving $\int \cos^2\theta . d\theta$

(b) Use $\cos^2\theta = 1 - \sin^2\theta$ to find $\int \sin^2\theta \, d\theta$

(c) Compare your answer with Exercise 3, § 8.0.