Section 10.2
Limits

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Definition of a limit

Example

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave at $x = 1$?

Solution:

$$f(1) = \frac{1^2 - 1}{1 - 1} = \frac{0}{0} \quad \text{undefined!}$$

Hence, we cannot substitute directly with $x = 1$, so instead we check values that are *very much close* to $x = 1$ and we check the corresponding values of $f(x)$.

x	0.9	0.99	0.9999	1	1.00001	1.001	1.01
$f(x)$				-			

Continue...

So we have seen that as x approaches 1, $f(x)$ approaches 2, we write

$$\lim_{x \to 1} f(x) = 2$$

To find the limit $\lim_{x \to a} f(x)$, we have

1. Substitute directly by $x = a$ in $f(x)$. If you get a real number, then that is the limit.

2. If you get undefined values such as $\frac{0}{0}$, we use *algebraic method* to clear any problem.

3. If you get values such as $\frac{\text{non–zero number}}{0}$, then we stop and we say that the limit does not exist (DNE).

Zero denominator of a rational function

A - Eliminating zero denominator by canceling common factor in the numerator and denominator ($\frac{0}{0}$ form).

Example

Find

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

Solution: Direct substitution gives

$$\frac{5^2 - 25}{5 - 5} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\lim_{x \to 5} \frac{x^2 - 5}{x - 5} = \lim_{x \to 5} \frac{(x - 5)(x + 5)}{(x - 5)}$$

$$= \lim_{x \to 5} (x + 5)$$

$$= 5 + 5 = 10$$

Exercise

Find

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1}$$

Find

$$\lim_{y \to 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2}$$

Solution: Direct substitution gives

$$\frac{5(0)^3 + 8(0)^2}{3(0)^4 - 16(0)^2} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\lim_{y \to 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2} = \lim_{y \to 0} \frac{y^2(5y + 8)}{y^2(3y^2 - 16)}$$
$$= \lim_{y \to 0} \frac{5y + 8}{3y^2 - 16}$$
$$= \frac{5(0) + 8}{3(0)^2 - 16} = \frac{8}{-16} = \frac{-1}{2}$$

Example

Find
$$\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9}$$

Solution: Direct substitution gives

$$\frac{0^2 - 3(0)}{0^2 - 9} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9} = \lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(x + 3)}$$
$$= \lim_{x \to 3} \frac{x}{x + 3}$$
$$= \frac{3}{6} = \frac{1}{2}$$

Exercise

Find

$$\lim_{x \to 1} \frac{x^2 - 3x + 4}{x^2 - 6x + 7}$$

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Example

Find

$$\lim_{x \to -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3}$$

Solution: Direct substitution gives

$$\frac{2(0)^2 + 3(0) + 1}{(0)^2 - 2(0) - 3} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\lim_{x \to -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3} = \lim_{x \to -1} \frac{2(x + \frac{1}{2})(x + 1)}{(x - 3)(x + 1)}$$

$$= \lim_{x \to -1} \frac{2(x + \frac{1}{2})}{x - 3}$$

$$= \frac{2(-\frac{1}{2})}{-4} = \frac{1}{4}$$

Exercise

(Old Final Exam Question) Find

$$\lim_{x \to 2} \frac{x^3 - 8}{3x^2 - x - 10}$$

Exercise

Find

$$\lim_{x \to 1} \frac{x^2 - 3x + 4}{x^2 - 6x + 7}$$

Example

Find

$$\lim_{x \to 1} \frac{4x^5 - 4}{5x^2 - 5}$$

Solution: Direct substitution gives

$$\frac{4(0)^5 - 4}{5(0)^2 - 5} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\begin{aligned}
\lim_{x \to 1} \frac{4x^5 - 4}{5x^2 - 5} &= \lim_{x \to 1} \frac{4(x-1)(x^4 + x^3 + x^2 + x + 1)}{5(x-1)(x+1)} \\
&= \lim_{x \to 1} \frac{4(x^4 + x^3 + x^2 + x + 1)}{5(x+1)} \\
&= \frac{20}{10} = 2
\end{aligned}$$

Exercise

Find

$$\lim_{x \to 2} \frac{x^4 - 16}{x^3 - 8}$$

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Find

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 5x + 6}$$

Solution: Direct substitution gives

$$\frac{2 - 2}{(2)^2 - 5(2) + 6} = \frac{0}{0} \quad \text{undefined!}$$

So we factor both the denominator and numerator to cancel the common zero.

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{(x - 2)}{(x - 2)(x - 3)}$$

$$= \lim_{x \to 2} \frac{1}{(x - 3)}$$

$$= \frac{1}{0} \quad \text{Does Not Exist}$$

Exercise

(Old Exam Question) Find

$$\lim_{x \to 5} \frac{5x - 25}{x^2 - 25}$$

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Conjugate and multiply by 1

B - Eliminating zero denominator by multiplying with the conjugate.

Example

Find

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

Solution: Direct substitution gives

$$\frac{\sqrt{9} - 3}{9 - 9} = \frac{0}{0} \quad \text{undefined!}$$

Since there is a square root and we cannot factor anything, we multiply both the numerator and denominator with the conjugate of the numerator (the one that contains the square root).

Continue...

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{1}{(\sqrt{x} + 3)}$$

$$= \frac{1}{6}$$

Exercise

(Old Exam Question) Find

$$\lim_{x \to 9} \frac{\sqrt{x-5}-2}{x-9}$$

Exercise

(Old Final Exam Question) Find

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$$

Properties of Limits

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$.

1. $\lim_{x \to a} c = c$.

2. $\lim_{x \to a} x^n = a^n$.

3. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + K$.

4. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot K$.

5. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) = cL$.

6. $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{K}$ if $K \neq 0$.

7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$. If n is even, then L must be non–negative.

Summary: These properties are telling us we can substitute directly with the value of a if there no problem.

Exercise

(Old Final Exam Question) Find

$$\lim_{x \to -3} \frac{x+1}{x^2+9}$$