Section 11.2
Rules of Differentiation

Dr. Abdulla Eid

College of Science

MATHS 104: Mathematics for Business II

Definition of the derivative

Recall: As in the homework, we find that

1. $\frac{d}{dx}\left(c\right) = 0$.
2. $\frac{d}{dx}\left(x\right) = 1$.
3. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.
4. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$.

Next we want to find the derivative of the power function $f(x) = x^n$, for any non-negative integer.

We will use the following:

- $z^n - x^n =$
  $$(z-x)\underbrace{(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1})}_{n \text{ terms each of power } n-1}.$$
- $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$

Power Rule

Theorem

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Let $f(x) = x^n$, then

$$\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{z^n - x^n}{z - x} \\
&= \lim_{z \to x} \frac{(z-x)\underbrace{(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}_{n \text{ terms each of power } n-1}}{(z-x)} \\
&= \lim_{z \to x} \underbrace{(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1})}_{n \text{ terms each of power } n-1} \\
&= \underbrace{x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \cdots + x^2x^{n-3} + xx^{n-2} + x^{n-1})}_{n \text{ terms of power } n-1}
\end{aligned}$$

Continue...

$$= \lim_{z \to x} \underbrace{\left(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1}\right)}_{n \text{ terms each of power } n-1}$$

$$= \underbrace{x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \cdots + x^2x^{n-3} + xx^{n-2} + x^{n-1})}_{n \text{ terms of power } n-1}$$

$$= \underbrace{x^{n-1} + x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1} + x^{n-1})}_{n \text{ terms of power } n-1}$$

$$= nx^{n-1}$$

Example

1. $\frac{d}{dx}\left(x^5\right) = 5x^4.$
2. $\frac{d}{dx}\left(x^2\right) = 2x.$
3. $\frac{d}{dx}\left(x^3\right) = 3x^2.$
4. $\frac{d}{dx}\left(x\right) = 1x^0 = 1.$
5. $\frac{d}{dx}\left(x^e\right) = ex^{e-1}.$
6. $\frac{d}{dx}\left(x^{-10}\right) = -10x^{-10-1} = -10x^{-11}.$
7. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{1}{2}}.$
8. $\frac{d}{dx}\left(\sqrt[6]{x^7}\right) = \frac{d}{dx}\left(x^{\frac{7}{6}}\right) = -\frac{7}{6}x^{\frac{1}{6}}.$
9. $\frac{d}{dx}\left(\frac{1}{x^3\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{1}{x^{\frac{7}{2}}}\right) = \frac{d}{dx}\left(x^{-\frac{7}{2}}\right) = -\frac{7}{2}x^{-\frac{7}{2}-1} = -\frac{7}{2}x^{-\frac{-9}{2}}.$

Constant Factor Rule

Theorem

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right)$$

Let $F(x) = cf(x)$.

$$\begin{aligned}
\frac{d}{dx}\left(c \cdot f(x)\right) = F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{c \cdot f(x+h) - (c \cdot f(x))}{h} \\
&= \lim_{h\to 0} \frac{c \cdot f(x+h) - f(x)}{h} \\
&= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= c\frac{d}{dx}\left(f(x)\right)
\end{aligned}$$

Example

1. $\frac{d}{dx}\left(5x^2\right) = 5\frac{d}{dx}\left(x^2\right) = 8 \cdot 2x = 10x.$
2. $\frac{d}{dx}\left(\frac{8}{x^5}\right) = 8\frac{d}{dx}\left(\frac{1}{x^5}\right) = 8\frac{d}{dx}\left(x^{-5}\right) = -40x^{-6}.$
3. $\frac{d}{dx}\left(7x^3\sqrt[4]{x}\right) = 7\frac{d}{dx}\left(x^{\frac{13}{4}}\right) = -\frac{7 \cdot 13}{4}x^{-\frac{9}{4}}.$

Sum Rule

Theorem

$$\frac{d}{dx}\left(f(x)+g(x)\right) = c\frac{d}{dx}\left(f(x)+\frac{d}{dx}\left(g(x)\right)\right)$$

Let $F(x) = f(x) + g(x)$.

$$\begin{aligned}
\frac{d}{dx}\left(f(x)+g(x)\right) = F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}\left(f(x)\right)+\frac{d}{dx}\left(g(x)\right)
\end{aligned}$$

Example

(Old Exam Question) Find y' and simplify:

1. $y = \frac{x^3}{3} - \frac{2}{x^2}$.
2. $y = x^2(4x+6)$.
3. $y = \frac{x^8+x^5}{x^2}$.
4. $y = \sqrt{2} + e^{\sqrt{2}} + \ln\sqrt{2}$.
5. $x^3 - \ln 2$.

Solution:

1. $y = \frac{1}{3}x^3 - 2x^{-2} \rightarrow y' = \frac{3}{3}x^2 + 4x^{-3} = x^2 + \frac{4}{x^3}$.
2. $y = 4x^3 + 6x^2 \rightarrow y' = 12x^2 + 12x$.
3. $y = x^6 + x^3 \rightarrow y' = 6x^5 + 3x^2$.
4. $y' = 0$.since all functions are constant functions.
5. $y' = 3x^2$.

Example

(Old Exam Question) Find all the points on the curve $y = x^3 - 3x + 6$ where the slope of the tangent line is 9.

Solution: Recall that the slope of the tangent line is the derivative, we need to find the derivative and make it equal to 9.

$$\begin{aligned}
\text{Slope of the tangent line} &= 9 \\
f'(x) &= 9 \\
3x^2 - 3 &= 9 \\
3x^2 - 12 &= 0 \\
x &= 2 \text{ or } x = -2 \text{ by the formula in Section 0.8}
\end{aligned}$$

The points are then

$$(2, 8) \text{ and } (-2, 4)$$

Example

(Old Final Exam Question) Find an equation of the tangent line to the curve $f(x) = 5 - \sqrt[4]{x}$ at $x = 1$.

Solution: Recall that the slope of the tangent line is the derivative. so we have

$$f(x) = 5 - x^{\frac{1}{4}} \to f'(x) = -\frac{1}{4}x^{-\frac{5}{4}}$$
$$m = f'(1) = \frac{1}{4}$$

Now $x_1 = 1$ and $y_1 = f(1) = 4$. Hence the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$
$$y - 4 = \frac{1}{4}(x - 1)$$
$$4y - 16 = x - 1$$
$$4y - x = 15$$

Exercise

(Old Final Exam Question) Find an equation of the tangent line to the curve $f(x) = \frac{\sqrt{x}(2-x^2)}{x}$ at $x = 4$.