Section 11.4 The Product Rule and the Quotient Rule

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MATHS 104: Mathematics for Business II

Motivation

Goal: We want to derive rules to find the derivative of product f(x)g(x) and quotient $\frac{f(x)}{g(x)}$ of two functions.

Example

We want to find (in a general way) the derivative of the functions:

•
$$f(x) = (3x+1)(5x+2).$$

• $f(x) = xe^{x}.$
• $f(x) = x^{2} \ln x.$
• $f(x) = \frac{3x+1}{x^{3}+2x+1}.$
• $f(x) = \frac{\ln x}{e^{x}+6x-3}.$

The Product Rule

Theorem

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

(f(x)g(x))' = (derivative of first)(second) + (first)(derivative of second)

Before we prove this theorem, recall that the definition of the derivative is

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
 and $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

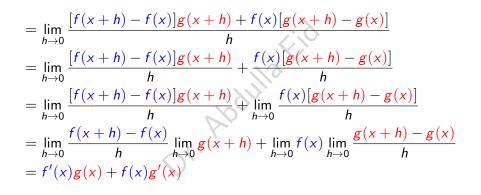
Proof: Let F(x) = f(x)g(x). Then,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We will use a "trick" by adding and subtracting f(x)g(x+h) in the middle of the numerator.

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}$$

Continue...



Example

Find the derivative of each of the following:

•
$$F(x) = (x^2 + 5x - 6)(6x^2 - 5x + 6)$$

• $F(x) = 2(\sqrt{x} + 5x - 3)(\sqrt[4]{x} - 4\sqrt{x})$

Solution: (1)

F'(x) = (derivative of first) (second) + (first)(derivative of second) $= (2x+5)(6x^2 - 5x + 6) + (x^2 + 5x - 6)(12x - 5)$

(2)

F'(x) = (derivative of first) (second) + (first) (derivative of second) $= 2 \left[(\frac{1}{2\sqrt{x}} + 5)(\sqrt[4]{x} - 4\sqrt{x}) + (\sqrt{x} + 5x - 3)(\frac{1}{4}x^{\frac{-3}{4}} - \frac{4}{2\sqrt{x}}) \right]$

Product rule for 3 functions

$$(fg)' = f'g + fg'$$

$$(fgh)' = f'gh + fg'h + fgh'$$

Example

Find the derivative of each of the following:

•
$$F(x) = (x-1)(x-2)(x^2-4)$$

Solution:

$$\bigcirc$$

$$F'(x) = (1)(x-2)(x^2-4) + (x-1)(1)(x^2-4) + (x-1)(x-2)(2x)$$

The Quotient Rule

Theorem

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

(denominator)derivative of numerator – (numerator)(derivative of denomina (denominator)²

To prove this theorem, we will use the product rule.

Proof: Let $F(x) = \frac{f(x)}{g(x)}$. We want to find F'(x). For that we apply the product rule to

$$F(x)g(x) = f(x)$$

(derivative of first)(second) + (first)(derivative of second) = f'(x)F'(x)g(x) + F(x)g'(x) = f'(x)F'(x)g(x) = f'(x) - F(x)g'(x) $F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$ $F'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$ $F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

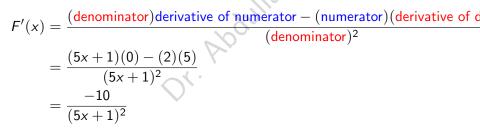
Example

(Old Exam Question) Find the derivative of each of the following:

1
$$F(x) = \frac{2}{5x+1}$$

2 $F(x) = \frac{1-x}{1-x^3}$

Solution: (1)



Continue...

Recall we want to find the derivative of $F(x) = \frac{1-x}{1-x^3}$.

 $F'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of of (denominator}))}{(\text{denominator})^2}$ $= \frac{(1-x^3)(-1) - (1-x)(-3x^2)}{(1-x^3)^2}$ $= \frac{-1+x^3+x^2-3x^3}{(1-x^3)^2}$ $= \frac{-1+x^2-2x^3}{(1-x^3)^2}$

Example

(Old Exam Question) The revenue of selling q units per month is given by $r(q) = \frac{500q}{q+16}$. Find the marginal revenue at q = 4.

Solution:

Marginal revenue = $\frac{dr}{da}$ (denominator)(derivative of numerator) – (numerator)(derivative of deno (denominator)² $=\frac{(q+16)(500)-(500q)(1)}{(q+16)^2}$ $=\frac{8000}{(a+16)^2}$ At q = 4, we have $\frac{dr}{dq}_{q=4} = \frac{8000}{(4+16)^2} = 20$