Section 11.4
The Product Rule and the Quotient Rule

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MATHS 104: Mathematics for Business II

Motivation

Goal: We want to derive rules to find the derivative of product $f(x)g(x)$ and quotient $\frac{f(x)}{g(x)}$ of two functions.

Example

We want to find (in a general way) the derivative of the functions:

- $f(x) = (3x + 1)(5x + 2)$.
- $f(x) = xe^x$.
- $f(x) = x^2 \ln x$.
- $f(x) = \frac{3x+1}{x^3+2x+1}$.
- $f(x) = \frac{\ln x}{e^x+6x-3}$.

The Product Rule

Theorem

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

$$(f(x)g(x))' = (\textit{derivative of first})(\textit{second}) + (\textit{first})(\textit{derivative of second})$$

Before we prove this theorem, recall that the definition of the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ and } g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Proof: Let $F(x) = f(x)g(x)$. Then,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We will use a "trick" by adding and subtracting $f(x)g(x+h)$ in the middle of the numerator.

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}$$

Continue...

$$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x+h)+f(x)[g(x+h)-g(x)]}{h}$$
$$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x+h)}{h} + \frac{f(x)[g(x+h)-g(x)]}{h}$$
$$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h)-g(x)]}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$
$$= f'(x)g(x) + f(x)g'(x)$$

Example

Find the derivative of each of the following:

1. $F(x) = (x^2 + 5x - 6)(6x^2 - 5x + 6)$
2. $F(x) = 2(\sqrt{x} + 5x - 3)(\sqrt[4]{x} - 4\sqrt{x})$

Solution: (1)

$$\begin{aligned} F'(x) &= (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second}) \\ &= (2x+5)(6x^2-5x+6) + (x^2+5x-6)(12x-5) \end{aligned}$$

(2)

$$\begin{aligned} F'(x) &= (\text{derivative of first})\,(\text{second}) + (\text{first})(\text{derivative of second}) \\ &= 2\left[(\frac{1}{2\sqrt{x}} + 5)(\sqrt[4]{x} - 4\sqrt{x}) + (\sqrt{x} + 5x - 3)(\frac{1}{4}x^{\frac{-3}{4}} - \frac{4}{2\sqrt{x}})\right] \end{aligned}$$

Product rule for 3 functions

$$(fg)' = f'g + fg'$$

$$(fgh)' = f'gh + fg'h + fgh'$$

Example

Find the derivative of each of the following:

1. $F(x) = (x-1)(x-2)(x^2-4)$

Solution:

$$F'(x) = (1)(x-2)(x^2-4) + (x-1)(1)(x^2-4) + (x-1)(x-2)(2x)$$

The Quotient Rule

Theorem

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

$$\frac{(\textit{denominator})\textit{derivative of numerator} - (\textit{numerator})(\textit{derivative of denomina}}{(\textit{denominator})^2}$$

To prove this theorem, we will use the product rule.

Proof: Let $F(x) = \frac{f(x)}{g(x)}$. We want to find $F'(x)$. For that we apply the product rule to

$$F(x)g(x) = f(x)$$

$$\text{(derivative of first)(second)} + \text{(first)(derivative of second)} = f'(x)$$

$$F'(x)g(x) + F(x)g'(x) = f'(x)$$

$$F'(x)g(x) = f'(x) - F(x)g'(x)$$

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$$

$$F'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Example

(Old Exam Question) Find the derivative of each of the following:

1. $F(x) = \frac{2}{5x+1}$
2. $F(x) = \frac{1-x}{1-x^3}$

Solution: (1)

$$F'(x) = \frac{(\text{denominator})\text{derivative of numerator} - (\text{numerator})(\text{derivative of d}}{(\text{denominator})^2}$$

$$= \frac{(5x+1)(0) - (2)(5)}{(5x+1)^2}$$

$$= \frac{-10}{(5x+1)^2}$$

Continue...

Recall we want to find the derivative of $F(x) = \frac{1-x}{1-x^3}$.

$$F'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of}}{(\text{denominator})^2}$$

$$= \frac{(1-x^3)(-1) - (1-x)(-3x^2)}{(1-x^3)^2}$$

$$= \frac{-1 + x^3 + x^2 - 3x^3}{(1-x^3)^2}$$

$$= \frac{-1 + x^2 - 2x^3}{(1-x^3)^2}$$

Example

(Old Exam Question) The revenue of selling q units per month is given by $r(q) = \frac{500q}{q+16}$. Find the marginal revenue at $q = 4$.

Solution:

$$\text{Marginal revenue} = \frac{dr}{dq}$$

$$= \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative of deno}}{(\text{denominator})^2}$$

$$= \frac{(q+16)(500) - (500q)(1)}{(q+16)^2}$$

$$= \frac{8000}{(q+16)^2}$$

At $q = 4$, we have

$$\left.\frac{dr}{dq}\right|_{q=4} = \frac{8000}{(4+16)^2} = 20$$