Section 12.1
The Derivative of Exponential and Logarithmic Functions

Dr. Abdulla Eid

College of Science

MATHS 104: Mathematics for Business II

Motivation

Goal: We want to derive rules to find the derivative of the functions $\ln x$ and e^x. We will use these results to find the derivative of $\log_a x$ and a^x.

The Derivative of ln

$$
\begin{aligned}
(\ln x)' &= \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h} \\
&= \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{h}\ln\left(\frac{x+h}{x}\right)\right) \\
&= \lim_{h\to 0} \left(\frac{1}{x}\cdot\frac{x}{h}\ln\left(1+\frac{h}{x}\right)\right) \\
&= \frac{1}{x}\cdot \lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}} \\
&= \frac{1}{x}\cdot \ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right) \\
&= \frac{1}{x}\cdot \ln\left(\lim_{k\to 0}(1+k)^{\frac{1}{k}}\right)
\end{aligned}
$$

Continue

$$
\begin{aligned}
(\ln x)' &= \frac{1}{x} \cdot \ln\left(\lim_{k\to 0}(1+k)^{\frac{1}{k}}\right) \\
&= \frac{1}{x} \cdot \ln(e) \\
&= \frac{1}{x}
\end{aligned}
$$

Theorem

The derivative of ln

$$\frac{d}{dx}\ln(u) = \frac{1}{u} \cdot u' = \frac{u'}{u}$$

Derivative of logarithm to any base

To find the derivative of $\log_a x$, we have to change it to ln.

$$
\begin{aligned}
(\log_a x)' &= \left(\frac{\ln x}{\ln a}\right)' \\
&= \frac{1}{\ln a}(\ln x)' \\
&= \frac{1}{x \ln a}
\end{aligned}
$$

Derivative of the exponential function

Goal: To find the derivative of e^x. We will use the chain rule and the fact that the logarithmic and exponential function are inverse to each other.

$$\ln(e^x) = x$$

Now take the derivative of both sides, we get

$$\frac{1}{e^x} \cdot (e^x)' = 1$$
$$(e^x)' = e^x$$

Theorem

The derivative of the exponential function

$$\frac{d}{dx}(e^u) = e^u \cdot u'$$

Derivative of any exponential

To find the derivative of a^x, we have to change it to natural exponential and we use the chain rule.

$$
\begin{aligned}
(a^x)' &= \left(e^{\ln a^x}\right)' \\
&= \left(e^{x \ln a}\right)' \\
&= \left(e^{x \ln a}\right)' \\
&= e^{x \ln a} \cdot \ln a \\
&= a^x \cdot \ln a
\end{aligned}
$$

The Derivative of an inverse Function

If we know the derivative of $f(x)$, we can use the chain rule to find the derivative of the inverse function $f^{-1}(x)$ as follows. Recall that

$$f(f^{-1}(x)) = x$$

Now take the derivative of both sides, we get

$$\begin{aligned} f'(f^{-1}(x)) \cdot (f^{-1}(x))' &= 1 \\ (f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$