Section 12.4
Implicit Differentiation

Dr. Abdulla Eid

College of Science

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Implicitly Defined Functions

So far, we have seen explicitly defined functions, i.e., functions of the form $y = f(x)$, where y is expressed totally in terms of x. Now we want to deal with implicitly defined functions such as the following:

Example

1. $x^2 + y^2 = 4$.
2. $e^{xy} - x^2 + 4y = 5x - 4$.
3. $\ln(y) + e^{2x} = y^2 e^{-x}$.

The above functions are said to be implicitly defined function.
Goal: To find the derivative y' of implicitly defined functions.

Example

Find y' for

$$y^2 = x$$

Solution 1:
We write y in term of x, so we get $y = \pm\sqrt{x}$. Hence,

$$y' = \pm\frac{1}{2\sqrt{x}}$$

Solution 2: To find the derivative y', we differentiate both sides with respect to x to get

$$2yy' = 1$$

$$y' = \frac{1}{2y}$$

Example

Find y' for

$$x^2 + y^2 = 4$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} 2x + 2yy' &= 0 \\ 2yy' &= -2x \\ y' &= \frac{-2x}{2y} \\ y' &= \frac{-x}{y} \end{aligned}$$

Example

(Old Exam Question) Find y' for

$$xy^2 - 6x = 5 + 2y$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} y^2 + 2xyy' - 6 - 2y' &= 0 \\ 2xyy' - 2y' &= -y^2 + 6 \\ (2xy - 2)y' &= -y^2 + 6 \\ y' &= \frac{-y^2 + 6}{2xy - 2} \end{aligned}$$

Example

(Old Exam Question) Find y' for

$$x + e^y = y + e^x$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} 1 + e^y \cdot y' &= y' + e^x \\ e^y y' - y' &= e^x - 1 \\ (e^y - 1)y' &= e^x - 1 \\ y' &= \frac{e^x - 1}{e^y - 1} \end{aligned}$$

Example

(Old Final Exam Question) Find y' for

$$(2x+y)^2 = x$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$2(2x+y)\cdot(2+y') = 1$$

$$(2+y') = \frac{1}{2(2x+y)}$$

$$y' = \frac{1}{2(2x+y)} - 2$$

Example

(Old Final Exam Question) Find y' for

$$xy^2 - y = x^2 + 1$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} y^2 + x(2yy') &= 2x \\ x(2yy') &= 2x - y^2 \\ y' &= \frac{2x - y^2}{2xy} \end{aligned}$$

Exercise

(Old Final Exam Question) Find y' for

$$x^2 + y^2 = 10y + 3xy$$

Example

(Old Exam Question) Find y' for

$$x = 7^{3y}$$

Solution: To find the derivative y', we differentiate both sides with respect to x to get

$$1 = 7^{3y} \ln 7 \cdot 3y'$$

$$y' = \frac{1}{3 \cdot 7^{3y} \ln 7}$$

Example

(Old Exam Question) Find the slope of the tangent line to the curve $e^{xy} + 2y = 7 + x$ at the point $(0, 3)$

Solution: To find the slope, we need to find the derivative y', so we differentiate both sides with respect to x to get

$$\begin{aligned} e^{xy}(y + xy') + 2y' &= 0 + 1 \\ e^{0\cdot 3}(3 + 0y') + 2y' &= 1 \\ 3 + 2y' &= 1 \\ 2y' &= -2 \\ y' &= -1 \end{aligned}$$

Example

(The Folium of Descarte')

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find an equation of the tangent line at $(3, 3)$.

Solution: (a) To find the derivative y', we differentiate both sides with respect to x to get

$$\begin{aligned} 3x^2 + 3y^2 y' &= 6y + 6xy' \\ 3y^2 y' - 6xy' &= 6y - 3x^2 \\ (3y^2 - 6x)y' &= 6y - 3x^2 \\ y' &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

Continue...

Example

(The Folium of Descarte')

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find an equation of the tangent line at $(3, 3)$.

Solution: (b) The equation of the tangent line is given by

$$y - y_1 = m(x - x_1)$$

We find the slope at $(3, 3)$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$m = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)}$$

$$m = -1$$

Continue...

Hence we have

$$
\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - 3 &= -(x - 3)\\
y + x &= 6
\end{aligned}
$$