Section 12.5
Logarithmic Differentiation

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MATHS 104: Mathematics for Business II

Logarithmic Differentiation

Goal: To find the derivative of $y = f(x)$, where $f(x)$ is possibly involving quotient, product, powers, etc.

Example

1. $y = \frac{(x+1)^4(3x^2+5)}{(4x-5)\sqrt[4]{4x^2+5}}$.
2. $y = \left(\frac{(x+5)(4x-2)^7}{x^2+5x+2}\right)^5$.
3. $y = x^{\sqrt{x}}$. — $\text{variable}^{\text{variable}}$.
4. $y = \ln x^{x^2+3x+5}$.

Idea

To differentiate $y = f(x)$,

1. Take the natural logarithm of both sides to get

$$\ln y = \ln\left(f(x)\right)$$

2. Simplify $\ln(f(x))$ by using the properties of the logarithms.
3. Differentiate both sided with respect to x.
4. Solve for y'.
5. Express the answer in term of x (substitute $f(x)$ for y).

Example

(Old Exam Question) Find y' for

$$y = \sqrt{\frac{5-4x}{1+x^2}}$$

Solution:
We take ln of both sides to get

$$\ln y = \ln\left(\sqrt{\frac{5-4x}{1+x^2}}\right)$$

We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln\left(\sqrt{\frac{5-4x}{1+x^2}}\right)$$

$$\ln y = \ln\left(\frac{5-4x}{1+x^2}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{5-4x}{1+x^2}\right)$$

$$\ln y = \frac{1}{2}\ \left(\ln(5-4x) - \ln(1+x^2)\right)$$

Continue...

Now we differentiate both sides and we solve for y'.

$$
\begin{aligned}
\ln y &= \frac{1}{2}\ \left(\ln(5-4x) - \ln(1+x^2)\right) \\
\frac{1}{y}y' &= \frac{1}{2}\ \left(\frac{-4}{5-4x} - \frac{2x}{1+x^2}\right) \\
y' &= y \cdot \frac{1}{2}\ \left(\frac{-4}{5-4x} - \frac{2x}{1+x^2}\right) \\
y' &= \sqrt{\frac{5-4x}{1+x^2}} \cdot \frac{1}{2}\ \left(\frac{-4}{5-4x} - \frac{2x}{1+x^2}\right)
\end{aligned}
$$

Example

Find y' for

$$y = \frac{(1-2x)^3(4+5x^6)^7}{\sqrt[3]{8-9x}}$$

Solution:
We take ln of both sides to get

$$\ln y = \ln\left(\frac{(1-2x)^3(4+5x^6)^7}{\sqrt[3]{8-9x}}\right)$$

We simplify the right hand side using the properties of logarithms to get

$$\begin{aligned}
\ln y &= \ln\left(\frac{(1-2x)^3(4+5x^6)^7}{\sqrt[3]{8-9x}}\right)\\
\ln y &= \ln(1-2x)^3 + \ln(4+5x^6)^7 - \ln\sqrt[3]{8-9x}\\
&= 3\ln(1-2x) + 7\ln(4+5x^6) - \frac{1}{3}\ln(8-9x)
\end{aligned}$$

Continue...

Now we differentiate both sides and we solve for y'.

$$\ln y = 3\ln(1-2x) + 7\ln(4+5x^6) - \frac{1}{3}\ln(8-9x)$$

$$\frac{1}{y}y' = \frac{-6}{1-2x} + \frac{210x^5}{4+5x^6} - \frac{-9}{3(8-9x)}$$

$$y' = y \cdot \left(\frac{-6}{1-2x} + \frac{210x^5}{4+5x^6} - \frac{-9}{3(8-9x)}\right)$$

$$y' = \frac{(1-2x)^3(4+5x^6)^7}{\sqrt[3]{8-9x}} \cdot \left(\frac{-6}{1-2x} + \frac{210x^5}{4+5x^6} - \frac{-9}{3(8-9x)}\right)$$

Example

(Old Final Exam Question) Find y' for

$$y = x^{x+1}$$

Solution:
We take ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln\left(x^{x+1}\right)$$
$$\ln y = (x+1)\ln x$$

Now we differentiate both sides and we solve for y'.

$$\frac{1}{y}y' = (1)(\ln x) + (x+1)\left(\frac{1}{x}\right)$$
$$y' = y \cdot \left(\ln x + \left(\frac{x+1}{x}\right)\right)$$
$$y' = x^{x+1} \cdot \left(\ln x + \left(\frac{x+1}{x}\right)\right)$$

Example

(Old Final Exam Question) Find y' for

$$y = (2x)^{1-x}$$

Solution:

We take ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln\left((2x)^{1-x}\right)$$
$$\ln y = (1-x)\ln(2x)$$

Now we differentiate both sides and we solve for y'.

$$\frac{1}{y}y' = (-1)(\ln(2x)) + (1-x)\left(\frac{1}{2x}\cdot 2\right)$$
$$y' = y\cdot\left(-\ln(2x) + \left(\frac{1-x}{x}\right)\right)$$
$$y' = (2x)^{1-x}\cdot\left(-\ln(2x) + \left(\frac{1-x}{x}\right)\right)$$

Example

(Old Final Exam Question) Find y' for

$$y = (3x)^{\sqrt{x}}$$

Solution:
We take ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln\left((3x)^{\sqrt{x}}\right)$$
$$\ln y = \sqrt{x}\ln(3x)$$

Now we differentiate both sides and we solve for y'.

$$\frac{1}{y}y' = \left(\frac{1}{2\sqrt{x}}\right)(\ln(3x)) + (\sqrt{x})\left(\frac{1}{3x}\cdot 3\right)$$
$$y' = y\cdot\left(\frac{\ln(3x)}{2\sqrt{x}} + \left(\frac{\sqrt{x}}{x}\right)\right)$$
$$y' = (2x)^{1-x}\cdot\left(\frac{\ln(3x)}{2\sqrt{x}} + \left(\frac{\sqrt{x}}{x}\right)\right)$$

Exercise

Find y' for

$$y = (\ln x)^{\ln x}$$

Example

(General Form) Find y' for

$$y = f(x)^{g(x)}$$

Solution:

We take ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln\left(f(x)^{g(x)}\right)$$
$$\ln y = g(x)\ln f(x)$$

Now we differentiate both sides and we solve for y'.

$$\frac{1}{y}y' = (g'(x))(\ln f(x)) + (g(x))\left(\frac{f'(x)}{f(x)}\right)$$
$$y' = y\cdot\left[(g'(x))(\ln f(x)) + (g(x))\left(\frac{f'(x)}{f(x)}\right)\right]$$
$$y' = f(x)^{g(x)}\cdot\left[(g'(x))(\ln f(x)) + (g(x))\left(\frac{f'(x)}{f(x)}\right)\right]$$