Section 14.9
Area between curves
2 Lectures

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MATHS 104: Mathematics for Business II

The Area between curves

Idea: To find the area between curves we have to do the following:

1. Sketch the graph of each curve.
2. Find the intersection points.
3. Integrate

$$\int_a^b (\text{top} - \text{bottom})\, dx$$

Example 1

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = 17 - x^2$, $y = x^2 - 1$.

Solution: First we find the intersection points, so have

$$\begin{aligned} 17 - x^2 &= x^2 - 1 \\ 18 &= 2x^2 \\ 2x^2 - 18 &= 0 \\ x &= -3 \text{ or } x = 3 \end{aligned}$$

$$
\begin{aligned}
\text{Area} &= \int_a^b (\text{top} - \text{bottom})\, dx \\
\text{Area} &= \int_{-3}^{3} \left((17 - x^2) - (x^2 - 1) \right) dx \\
&= \int_{-3}^{3} (18 - 2x^2)\, dx \\
&= \left[18x - \frac{2}{3}x^3 \right]_{-3}^{3} \\
&= 72
\end{aligned}
$$

Example 2

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = x^2 + 1$, $y = 2x + 1$.

Solution: First we find the intersection points, so have

$$x^2 + 1 = 2x + 1$$
$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x = 0 \text{ or } x = 2$$

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

$$\begin{aligned}
\text{Area} &= \int_0^2 ((2x+1) - (x^2+1))\, dx \\
&= \int_0^2 (2x - x^2)\, dx \\
&= \left[x^2 - \frac{1}{3}x^3 \right]_0^2 \\
&= \frac{4}{3}
\end{aligned}$$

Example 3

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = e^x$ from $x = 1$ to $x = 3$.

Solution:

$$\begin{aligned}
\text{Area} &= \int_a^b (\text{top} - \text{bottom})\, dx \\
\text{Area} &= \int_1^3 (e^x)\, dx \\
&= \int_1^3 (e^x)\, dx \\
&= [e^x]_1^3 \\
&= e^3 - e^1
\end{aligned}$$

Example 4

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = 4 - x^2$, $y = 2x^2 + 1$.

Solution: First we find the intersection points, so have

$$4 - x^2 = 2x^2 + 1$$
$$3 = 3x^2$$
$$3x^2 - 1 = 0$$
$$x = -1 \text{ or } x = 1$$

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

$$\begin{aligned}
\text{Area} &= \int_{-1}^{1} \left((4 - x^2) - (2x^2 + 1) \right) dx \\
&= \int_{-1}^{1} (3 - 3x^2)\, dx \\
&= \left[3x - x^3 \right]_{-1}^{1} \\
&= 4
\end{aligned}$$

Example 5

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = x^2$, $y = \sqrt{x}$.

Solution: First we find the intersection points, so have

$$\begin{aligned} x^2 &= \sqrt{x} \\ x^4 &= x \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \\ x = 0 &\text{ or } x = 1 \end{aligned}$$

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

$$\text{Area} = \int_0^1 (\sqrt{x}) - (x^2)\, dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\, dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{3}$$

Example 6

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = x^2 + x - 2$ and x–axis.

Solution: Recall that the x–axis is represented by the equation $y = 0$. First we find the intersection points, so have

$$x^2 + x - 2 = 0$$
$$x^2 + x - 2 = 0$$
$$x = -2 \text{ or } x = 1$$

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

$$\begin{aligned}\text{Area} &= \int_{-2}^{1} ((0) - (x^2 + x - 2))\, dx \\ &= \int_{-2}^{1} (-x^2 - x + 2)\, dx \\ &= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1} \\ &= \frac{9}{2}\end{aligned}$$

Example 7

(Old Final Exam Question) Sketch and find the area of the region bounded by the curves $y = 17 - x^2$, $y = x^2 - 1$ from $x = -4$ to $x = 1$.

Solution: First we find the intersection points, so have

$$\begin{aligned} 17 - x^2 &= x^2 - 1 \\ 18 &= 2x^2 \\ 2x^2 - 18 &= 0 \\ x = -3 &\text{ or } x = 3 \end{aligned}$$

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

$$\text{Area} = \int_{-4}^{-3} ((x^2 - 1) - (17 - x^2))\, dx + \int_{-3}^{1} ((17 - x^2) - (x^2 - 1))\, dx$$

$$= \int_{-4}^{-3} (2x^2 - 18)\, dx + \int_{-3}^{1} (18 - 2x^2)\, dx$$

$$= \left[\frac{2}{3}x^3 - 18x\right]_{-4}^{-3} + \left[18x - \frac{2}{3}x^3\right]_{-3}^{1}$$

$$= 60$$