Section 1.1
System of Linear Equations

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MATHS 211: Linear Algebra

Goals:

1. Define system of linear equations and their solutions.
2. To represent system of linear equations by several ways.
3. To solve system of linear equations using Gaussian–Jordan Elimination.
4. To solve system of linear equations using the inverse of a matrix.
5. To solve system of linear equations using Cramer's rule.

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Linear equations

Definition 1

A **linear equation** in the variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + + a_2x_2 + + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, a_2, \dots, a_n$ are real numbers.

Note: A linear equations is of degree *one* in the variables.

Example 2

Which of the following are linear equations and why?

1. $4x_1 + 3x_2 = -6$
2. $x_1 + x_2 - 5 = x_3 + 2x_1$
3. $3x + 2y - z + w = 5$
4. $x_1 + x_2 = x_1x_2$
5. $x_2 = \sqrt{6}x_1 + x_3$
6. $x_2 = 6\sqrt{x_1} + x_3$

System of linear equations

Definition 3

A **system of linear equation** or (linear system) in the variables $x_1, x_2, \ldots, x_n$ is a finite collection of linear equations.

Example 4

$$\begin{aligned} -x_1 + x_2 &= 5 \\ x_1 + 5x_2 &= 1 \end{aligned} \qquad \begin{aligned} 2x - 7y &= -1 \\ x + 3y &= 6 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 + 3x_3 &= 8 \\ x_1 + 3x_2 - 2x_3 &= 7 \\ -3x_1 + x_3 &= 3 \end{aligned}$$

General definition of linear system

Definition 5

A **general linear system** of m equations and n variables $x_1, x_2, \ldots, x_n$ can be written as

$$a_{11}x_1 + + a_{12}x_2 + + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + + a_{22}x_2 + + \cdots + a_{2n}x_n = b_2$$

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$$a_{m1}x_1 + + a_{m2}x_2 + + \cdots + a_{mn}x_n = b_m$$

Solution to a system of linear equations

Definition 6

A **solution** is a list of numbers $(s_1, s_2, \ldots, s_n)$ that makes each equation a true statement when we substitute $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$. A **solution set** is the set of all possible solution to a linear system.

Example 7

Show that $(3, 2)$ is a solution to the system

$$
\begin{aligned}
3x_1 - x_2 &= 7 \\
-x_1 + 4x_2 &= 5
\end{aligned}
$$

Example 8

Show that $(-1, 0, 2)$ is a solution to the system

$$
\begin{aligned}
x + y - z &= 1 \\
3x + y &= -3
\end{aligned}
$$

Solution to a system of linear equations

Example 9

Show that $(1+5t, 3-t, t)$ is a solution to the system for any $t \in \mathbb{R}$.

$$\begin{aligned} x_1 + 6x_2 + x_3 &= 19 \\ x_1 - 5x_3 &= 1 \\ 3x_1 - x_2 + 16x_3 &= 0 \end{aligned}$$

The solution of linear system that depends on free variable is called **parametric solution**

Representing Linear System as augmented matrix and matrix form

Example 10

Represent the linear system in two forms

$$\begin{aligned} -x_1 + x_2 &= 5 \\ x_1 + 5x_2 &= 1 \end{aligned}$$

$$\begin{aligned} 2x - 7y &= -1 \\ x + 3y &= 6 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 + 3x_3 &= 8 \\ x_1 + 3x_2 - 2x_3 &= 7 \\ -3x_1 + x_3 &= 3 \end{aligned}$$

Solving System of Linear Equations using elementary row operations

Example 11

Solve the system

$$2x - 7y = -1$$
$$x + 3y = 6$$

Solution: First we write the **augmented matrix** of the system which is

$$\begin{pmatrix} 2 & -7 & | & -1 \\ 1 & 3 & | & 6 \end{pmatrix}, \qquad R_1 \leftrightarrow R_2$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 2 & -7 & | & -1 \end{pmatrix}, \qquad R_2 \to R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 2-2(1) & -7-2(3) & | & -1-2(6) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 2-2(1) & -7-2(3) & | & -1-2(6) \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 0 & -13 & | & -13 \end{pmatrix} \qquad R_2 \to \frac{1}{-13} R_2$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 0 & 1 & | & 1 \end{pmatrix} R_1 \to R_1 - 3R_2$$

$$\begin{pmatrix} 1-3(0) & 3-3(1) & | & 6-3(1) \\ 0 & 1 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 1 \end{pmatrix}$$

So $x = 3$ and $y = 1$ and thus the solution set is $\{(3,1)\}$

Example 12

Solve the system

$$\begin{aligned} x + y - z &= 7 \\ 4x + 6y - 4z &= 8 \\ x - y - 5z &= 23 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4 & 6 & -4 & 8 \\ 1 & -1 & -5 & 23 \end{array}\right), \quad R_2 \to R_2 - 4R_1 \quad R_3 \to R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4-4(1) & 6-4(1) & -4-4(-1) & 8-4(7) \\ 1-1 & -1-1 & -5-(-1) & 23-7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & 0 & -20 \\ 0 & -2 & -4 & 16 \end{array}\right) \qquad R_2 \to \frac{1}{2}R_2$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & -2 & -4 & | & 16 \end{pmatrix}, \qquad R_3 \to R_3 + 2R_2$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & -2+2(1) & -4+2(0) & | & 16+2(-10) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & -4 & | & -4 \end{pmatrix} R_3 \to \frac{1}{-4} R_3$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \qquad R_1 \to R_1 + R_3$$

$$\begin{pmatrix} 1+1(0) & 1+1(0) & -1+1(1) & | & 7+1(1) \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & | & 8 \\ 0 & 1 & 0 & | & -10 \end{pmatrix} \qquad R_1 \to R_1 - R_2$$

$$\begin{pmatrix} 1 & 1 & 0 & | & 8 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \qquad R_1 \to R_1 - R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 18 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

So $x = 18$, $y = -10$, and $z = 1$.
Solution Set $= \{(18, -10, 1)\}$.

Example 13

Solve the system

$$x + 4y = 9$$
$$3x - y = 6$$
$$2x - 2y = 4$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 2 & -2 & 4 \end{array}\right), \qquad R_2 \to R_2 - 3R_1 \quad R_3 \to R_3 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3-3(1) & -1-3(4) & 6-3(9) \\ 2-2(1) & -2-2(4) & 4-2(9) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -10 & -14 \end{array}\right) \qquad R_2 \to \frac{1}{-13} R_2$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & -10 & | & -14 \end{pmatrix}, \qquad R_3 \to R_3 + 10R_2$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & -10+10(1) & | & -14+10(\frac{-21}{-13}) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & 0 & | & \frac{28}{13} \end{pmatrix}$$

We have $0 = \frac{28}{13}$ which is a false statement and thus there will be no solution.

Example 14

Solve the system

$$
\begin{aligned}
x - y + 2z &= 5\\
2x - 2y + 4z &= 10\\
3x - 3yy + 6z &= 15
\end{aligned}
$$

Solution:

Example 15

Solve the system

$$x + y + z = 9$$
$$x + 5y + 10z = 44$$

Solution:

Types of solutions for a linear system

- Consistent. It has a solution.
  1. Unique solution. No free variables.
  2. Infinitely many solutions
- Inconsistent. It has no solution.

Example 16

Find the value of h such that the system has (a) unique solution, (b) no solution, (c) infinitely many solutions.

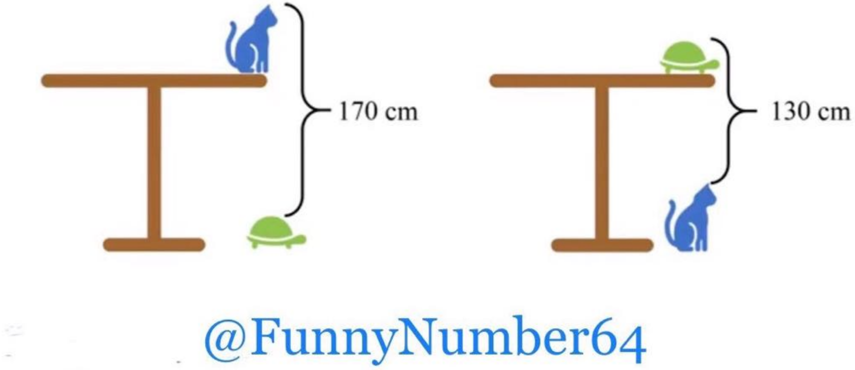
$$\begin{aligned} x + hy &= 4 \\ 3x + 6y &= 8 \end{aligned}$$

Example 17

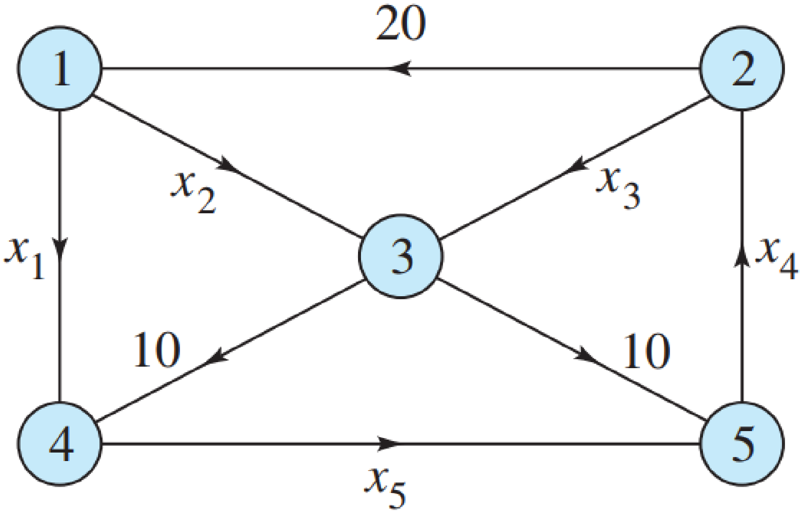
Find the value of h, k such that the system has (a) unique solution, (b) no solution, (c) infinitely many solutions.

$$x_1 + hy = 2$$
$$4x + 8y = k$$

What is the height of the table?



Network Flow

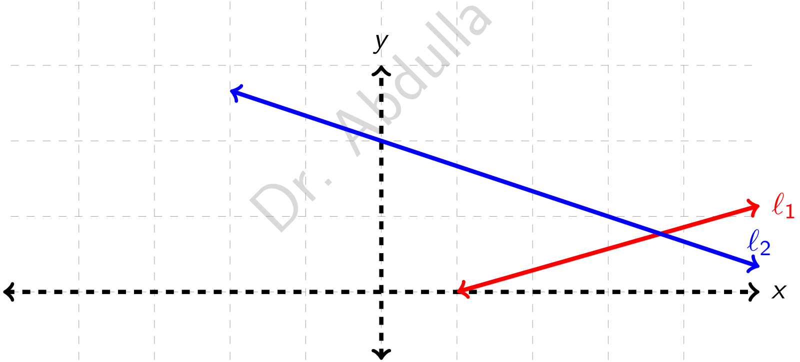


Lines in 2–D

Example 18

Solve the system geometrically

$$2x - 7y = 2, \qquad x + 3y = 6$$

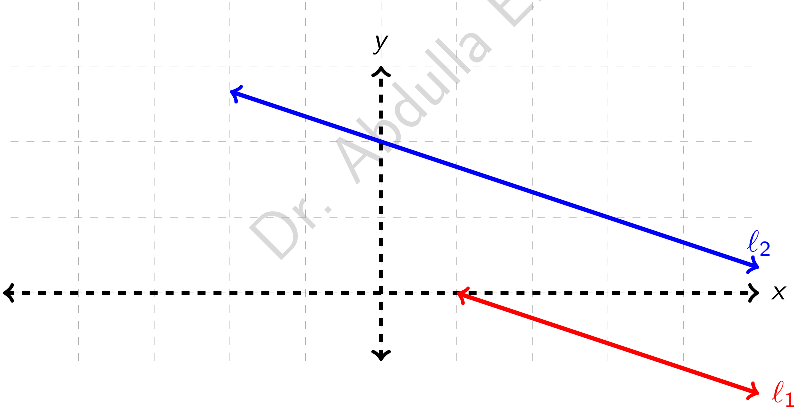


Lines in 2–D

Example 19

Solve the system geometrically

$$2x + 6y = 2, \qquad x + 3y = 6$$

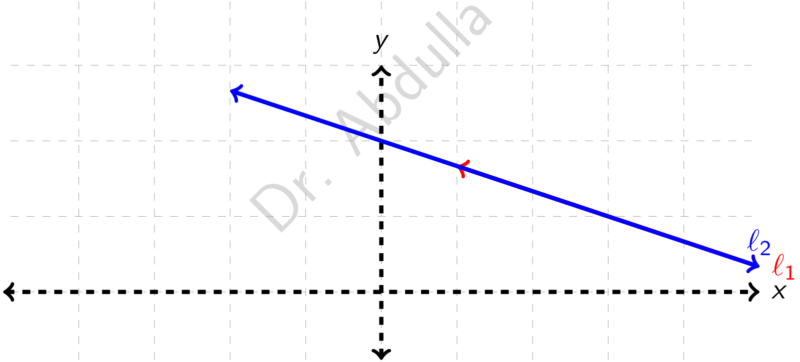


Lines in 2–D

Example 20

Solve the system geometrically

$$\color{red}{2x + 6y = 12}, \qquad \color{blue}{x + 3y = 6}$$

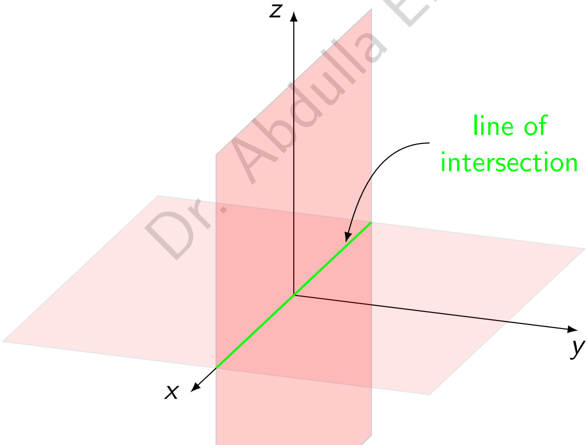


Lines in 3–D

Example 21

Solve the system geometrically

$$x + 2y - z = 3, \qquad x + 3y + 5z = -1$$



Solving Linear System using the inverse of a matrix

Example 22

Solve

$$3x + y = 2$$
$$4x + y = 3$$

Solution: This system can be written in a matrix multiplication form as

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$I_2 \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = {\color{blue}A^{-1}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise 23

Solve the following system using the inverse matrix method.

$$\begin{aligned} 2x - 3y &= 9 \\ 4x + y &= 1 \end{aligned}$$

Cramer's Rule

Theorem 24

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) =\neq 0$*, then the system has a unique solution given by*

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

Example 25

Solve using Cramer's rule the following system of linear equations

$$3x_1 + x_2 = 2$$
$$4x_1 + x_2 = 3$$

Solution:

Example 26

Solve using Cramer's rule the following system of linear equations

$$\begin{aligned} 3x_1 + 5x_2 &= 7 \\ 6x_1 + 2x_2 + 4x_3 &= 10 \\ -x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$

Solution:

The equation $A\mathbf{x} = \mathbf{b}$

Theorem 27

The following are equivalent:

1. *A is invertible.*
2. $\det(A) \neq 0$.
3. *The reduced row echelon form is I_n.*
4. $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.
5. $A\mathbf{x} = \mathbf{b}$ *has a unique solution for every* $n \times 1$ *matrix* $\mathbf{b}$.