Section 1.3 (Part 1)
Matrices

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MATHS 211: Linear Algebra

Goal

We want to learn

1. What a matrix is?
2. How to add or subtract two matrices?
3. How to multiple two matrices?
4. How to find the multiplicative inverse?
5. What is the determinant of a matrix and why it is useful?
6. How to solve system of linear equations using matrices?

1- Matrices

Definition 1

A **matrix** is just a rectangular array of entries. It is described by the **rows** and **columns**.

note: The work matrix is singular. The plural of matrix is *matrices* (pronounced as'may tri sees').

Example 2

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 & 1 \\ 7 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Usually the matrices are written in the form

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}, \quad \text{with } B_{\underbrace{i}_{\text{row}} \underbrace{j}_{\text{column}}}$$

Definition 3

An $m \times n$–**matrix** is a rectangular array consists of m rows and n columns.

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1n} \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ A_{n1} & A_{n2} & \cdot & \cdot & \cdot & A_{nn} \end{pmatrix} = (A_{ij})_{m \times n}$$

where A_{ij} is the entry in the row i and column j.

Example 4

Let

$$A = \begin{pmatrix} 3 & -2 & 7 & 3 \\ 2 & 1 & -1 & -5 \\ 4 & 3 & 2 & 1 \\ 0 & 8 & 0 & 2 \end{pmatrix}$$

1. What is the size of A?
2. Find $A_{21}, A_{42}, A_{32}, A_{34}, A_{44}, A_{55}$.
3. What are the entries of the second row?

Definition 5

If A is a matrix, the **transpose** of A is a new matrix A^T formed by interchanging the rows and the columns of A, i.e.,

$$A^T = (A_{ji})$$

Example 6

Find the transpose M^T and $(M^T)^T$.

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ 7 & 1 & 6 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 1 & 2 & 5 \end{pmatrix}$$

Solution:

1. $$A^T = \begin{pmatrix} 6 & 2 \\ -3 & 4 \end{pmatrix} \qquad \text{and} \qquad (A^T)^T = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$$

Example 7

Find the transpose M^T and $(M^T)^T$.

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ 7 & 1 & 6 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 1 & 2 & 5 \end{pmatrix}$$

Solution:

1. $$B^T = \begin{pmatrix} 2 & 7 \\ 1 & 1 \\ 3 & 6 \end{pmatrix} \quad \text{and} \quad (B^T)^T = \begin{pmatrix} 5 & 6 & 1 \\ 7 & 1 & 2 \end{pmatrix}$$

2. $$C^T = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 5 \end{pmatrix} \quad \text{and} \quad (C^T)^T = \begin{pmatrix} 3 & 1 & 2 & 5 \end{pmatrix}$$

Note:

1. $(A^T)^T = A$.
2. A matrix A is called **symmetric** if $A^T = A$.

Question: When two matrices are equal?

Definition 8

Two matrices A and B are *equal* if they have the same size and the same entries at the same position, i.e.,

$$A_{ij} = B_{ij}$$

Example 9

Solve the matrix equation

$$\begin{pmatrix} 4 & 2 & 1 \\ x & 2y & 3z \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ -3 & -8 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Solution:

$$x = -3, 2y = -8 \to y = -4, 3z = 0 \to z = 0$$

Special Matrices

- Zero matrix $\mathbf{0}_{m\times n} = (0)_{m\times n}$ “zero everywhere“.

$$\begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Square matrix if $m = n$ (having the same number of rows and columns).

$$\begin{pmatrix} 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 \\ 1 & -5 \end{pmatrix}, \quad \begin{pmatrix} 6 & 5 & -1 \\ 1 & 3 & 3 \\ 8 & -9 & 0 \end{pmatrix}$$

- Diagonal matrix if it is a square matrix ($m = n$) and all entries off the main diagonal are zeros.

$$\begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -11 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Special Matrices

- Upper Diagonal matrix if it has zeros below the main diagonal (entries are 'upper' the main diagonal).

$$\begin{pmatrix} 3 & 5 \\ 0 & -5 \end{pmatrix}, \quad \begin{pmatrix} 6 & 3 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 2 & -7 \\ 0 & -5 & -4 & 6 \\ 0 & 0 & -11 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

- Lower Diagonal matrix if it has zeros above the main diagonal (entries are 'lower' the main diagonal).

$$\begin{pmatrix} 3 & 0 \\ 7 & -5 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 7 & 6 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 6 & -5 & 0 & 0 \\ -8 & -4 & -11 & 0 \\ 1 & 4 & 7 & 2 \end{pmatrix}$$

Special Matrices

- Row vector is a matirx with only *one* row.

$$\begin{pmatrix} 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 5 & 13 & 12 \end{pmatrix}, \quad \begin{pmatrix} 7 & 3 & 0 & -2 & 6 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

- Column vector is a matrix with only *one* column.

$$\begin{pmatrix} 3 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 1 \\ 8 \\ 0 \end{pmatrix}$$

- Identity matrix I_n if $m = n$ and has one in the main diagonal and zero elsewhere.

$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition 10

If A is a square matrix, the **trace** of A is sum of the main diagonal

$$tr(A) = \sum_{i=1}^{n} A_{ii}$$

Example 11

Find the trace of the following matrices.

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ 7 & 1 & 6 \\ 0 & 0 & -3 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 1 & 2 & 5 \\ 13 & 0 & 22 & 15 \\ 33 & 41 & 13 & 65 \\ 34 & 15 & 2 & -10 \end{pmatrix}$$

Solution: