Section 4.1
Vector Spaces

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MATHS 211: Linear Algebra

Goal:

1. Define Vector spaces.
2. Explore important examples of vector spaces

Vectors

- In short, vectors are elements that we can add and multiply by a scalar.
- Let V be any arbitrary nonempty set of elements together with two operations $(+, \cdot)$ (**addition** and **multiplication by a scalar**).

$$
\begin{aligned}
+ : V \times V &\to V \\
(\mathbf{u}, \mathbf{v}) &\mapsto \mathbf{u} + \mathbf{v}
\end{aligned}
$$

and

$$
\begin{aligned}
\cdot : \mathbb{R} \times V &\to V \\
(k, \mathbf{u}) &\mapsto k \cdot \mathbf{u}
\end{aligned}
$$

We say $(V, +, \cdot, \mathbb{R})$ a real vector space if it satisfies the following axioms

Axioms

1. (Closure) If $\mathbf{u}, \mathbf{v}$ in V, then so $\mathbf{u} + \mathbf{v}$ in V.
2. (commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. (Associative) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. (Zero) There is an element $\mathbf{0}$ in V called the **zero vector** such that $\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u}$.
5. (Negative) For each $\mathbf{u}$, there is an element $(-\mathbf{u})$ in V called the **negative vector** of $\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0} = (-\mathbf{u}) + uu$.
6. If $k \in \mathbb{R}$ and $\mathbf{u} \in V$, then so $k \cdot \mathbf{u} \in V$.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.
8. $(k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$.
9. $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Premier Example, $\mathbb{R}^n$

Example 1

Consider the set

$$\mathbb{R}^n := \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

$$k\mathbf{u} = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Identification

We usually identify $\mathbb{R}^n$ with the set of all columns vector of length n, i.e.,

$$\mathbb{R}^n = \mathrm{Mat}(n, 1, \mathbb{R}) = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$$

Of course, we can identify $\mathbb{R}$ with the set of all row vectors of length n.

Zero Vector Space

Example 2

Consider the set

$$V := \{*\}$$

$$\mathbf{u} + \mathbf{v} = * + * = *$$

$$k\mathbf{u} = k \cdot * = *$$

This is called the *zero vector space* and usually we denoted by $V = \{\mathbf{0}\}$.

Premier Example, $\mathrm{Mat}(m, n, \mathbb{R})$

Example 3

Consider the set

$$\mathrm{Mat}(m, n, \mathbb{R}) := \{(a_{ij}) \mid a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$$

$$\mathbf{u} + \mathbf{v} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$
$$k\mathbf{u} = k(a_{ij}) = (ka_{ij})$$

Premier example, $\mathbb{P}_n$

Example 4

Consider the set

$$\mathbb{P}_n := \{a_0 + a_1X + a_2X^2 + \cdots + a_nX^n \mid a_i \in \mathbb{R}\}$$

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) + (b_0 + b_1X + b_2X^2 + \cdots + b_nX \\ &= (a_0 + b_0) + (a_1 + b_1)X + (a_2 + b_2)X^2 + \cdots + (a_n + b_n)X^n \\ k\mathbf{u} &= k(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n)\end{aligned}$$

Example, $\mathbb{R}^\infty$

Example 5

Consider the set

$$\mathbb{R}^\infty := \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots)$$
$$k\mathbf{u} = k(a_1, a_2, \dots) = (ka_1, ka_2, \dots)$$

The vector space of infinite sequences of real numbers.

Premier Example, Maps($\mathbb{R}, \mathbb{R}$)

Example 6

Consider the set

$$\text{Maps}(\mathbb{R}, \mathbb{R}) := \{f \mid f : \mathbb{R} \to \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (f + g)(x) = f(x) + g(x)$$
$$k\mathbf{u} = (kf)(x) = kf(x)$$

The vector space of all real valued function on $\mathbb{R}$.

Unusual Example, $\mathbb{R}$ with nonstandard operations

Example 7

Consider the set

$$\mathbb{R} := \{a \mid a \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = a + b = ab$$
$$k\mathbf{u} = ka = a^k$$

This defined indeed a vector space of real numbers. Why?

Non–Example, $\mathbb{R}^2$ with nonstandard operations

Example 8

Consider the set

$$\mathbb{R}^2 := \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$$

$$\mathbf{u} + \mathbf{v} = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$k\mathbf{u} = k(a_1, a_2) = (0, ka_2)$$

This is **not** a vector space of infinite sequences of real numbers. Why?

Properties of Vector spaces

Theorem 9

Let $(V, +, \cdot, \mathbb{R})$ *be a vector space,* $\mathbf{u} \in V$ *and* $k \in \mathbb{R}$*. Then*

1. $0\mathbf{u} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = (-\mathbf{u})$
4. *if* $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = \mathbf{0}$