Section 4.2
Subspaces

Dr. Abdulla Eid

College of Science

MATHS 211: Linear Algebra

Goal:

1. Define subspaces.
2. Subspace test.
3. Linear Combination of elements.
4. Subspace generated by elements (Span).

Subspace

Definition 1

Let V be a vector space. A subset W of V is called a **subspace** of V if W is itself a vector space under the same operations of V.

Subspace Test

Theorem 2

If W is a subset of V such that

1. $\mathbf{0} \in W$.
2. *For all* $\mathbf{u}, \mathbf{v} \in W$, *we have* $\mathbf{u} + \mathbf{v} \in W$.
   (Closed under addition — Axiom (1))
3. *For all* $\mathbf{u} \in W, k \in \mathbb{R}$, *we have* $k\mathbf{u} \in W$.
   (Closed under scalar multiplication — Axiom (6))

Then W is a subspace of V.

In short, we need to check that the zero is in W and W is closed under $+$ and $\cdot$

Zero Subspace

Example 3

Let V be any vector space. Let $W = \{\mathbf{0}\}$. Then W is a subspace of V.

We call W the zero subspace of V.

Lines through the origin

Example 4

Let m be a fixed real number. Consider the subset of $V = \mathbb{R}^2$

$$W := \left\{ \begin{pmatrix} x \\ mx \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

Then W is a subspace of $\mathbb{R}^2$

Example 5

Determine whether the following is a subspace of $\mathbb{R}^3$ or not.

$$W := \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R}, c = a - b \right\}$$

Example 6

Determine whether the following is a subspace of $\mathbb{R}^3$ or not.

$$W := \left\{ \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} \mid a \in \mathbb{R}, \right\}$$

Example 7

Determine whether the following is a subspace of $\mathrm{Mat}(n, n, \mathbb{R})$ or not.

$$W := \left\{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^T = -A\right\}$$

Example 8

Determine whether the following is a subspace of $\mathrm{Mat}(n, n, \mathbb{R})$ or not.

$$W := \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid tr(A) = 0\}$$

Example 9

Determine whether the following is a subspace of $\mathbb{P}_3$ or not.

$$W := \left\{ a_0 + a_1 X + a_2 X^2 + a_3 X^3 \,\middle|\, a_1 = a_2 \right\}$$

Example 10

Determine whether the following is a subspace of $\text{Maps}(\mathbb{R}, \mathbb{R})$ or not.

$$C(-\infty, \infty) := \{f \mid f \text{ is continuous }\}$$

$$C^1(-\infty, \infty) := \{f \mid f \text{ is differentiable }\}$$

$$C^2(-\infty, \infty) := \{f \mid f \text{ is twice differentiable }\}$$

$$C^\infty(-\infty, \infty) := \{f \mid f \text{ is infinitely many differentiable }\} \text{ — Smooth function}$$

Give an example of a function in $C^\infty(-\infty, \infty)$?

Intersection of subspaces

Theorem 11

Let W_1, W_2 be two subspaces of a vector space V. Then, the intersection of W_1 and W_2 is also a subspace of V.

Theorem 12

Let $W_1, W_2, \ldots, W_n$ be two subspaces of a vector space V. Then, the intersection of $W_1, W_2, \ldots, W_n$ is also a subspace of V.

Union of subspaces

Example 13

Let W_1, W_2 be two subspaces of a vector space $\mathbb{R}^2$ that are given by

$$W_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \qquad W_1 = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}$$

Verify that W_1, W_2 are subspaces but $W_1 \cup W_2$ is not.

Do HOMEWORK 1

Linear Combination

Definition 14

If w is a vector in a vector space V, then w is said to be a **linear combination** of the vectors $v_1, \ldots, v_n$ if w can be expressed in the form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n,$$

where $k_1, k_2, \ldots, k_n \in \mathbb{R}$ which are called the **coefficients** of the linear combination.

Example 15

Express the following as linear combination of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

1. $(6, 11, 6)$
2. $(7, 8, 9)$

Solution

Example 16

Let $\mathbf{u} = (1, -3, 2)$, $\mathbf{v} = (1, 0, -4)$. Determine whether the following is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

1. $(0, -3, 6)$
2. $(1, 6, -16)$

Example 17

Express the following as linear combination of $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$,

$B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix}$

1. $\begin{pmatrix} 2 & 5 \\ -2 & 4 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix}$

Solution

Example 18

Express the following as linear combination of $P_1 = 2 + X + 4x^2$, $P_2 = 1 - X + 3x^2$, and $P_3 = 3 + 2X + 5X^2$.

1. 0
2. $2 - X + 6X^2$

Solution

Subset generated by elements is a subspace

Theorem 19

If $S = \{v_1, v_2, \ldots, v_n\}$ is a nonempty set of vectors in a vector space V. Then,

1. *The set W of all possible linear combinations of vectors in S is a subspace, i.e.,*

$$W = \{k_1 v_1 + k_2 v_2 + \cdots + k_n v_n \mid k_1, k_2, \ldots, k_n \in \mathbb{R}\}$$

2. *The set W is the "smallest" subspace of V that contain all of the vectors in S in the sense of containment relationship.*

We denote the set W above by

$$W = \text{span}\{v_1, \ldots, v_n\} \text{ or } W = \text{span}(S) \text{ or } W = \langle v_1, \ldots, v_n \rangle$$

Proof

Recall: The equation $A\mathbf{x} = \mathbf{b}$

Theorem 20

The following are equivalent:

1. *A is invertible.*
2. $\det(A) \neq 0$.
3. *The reduced row echelon form is I_n.*
4. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*
5. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.*

Spanning set of the whole space

Example 21

Determine whether $\mathbf{v}_1 = (2, -1, 2)$, $\mathbf{v}_2 = (4, 1, 3)$, and $\mathbf{v}_3 = (2, 2, 1)$ span $\mathbb{R}^3$.

Solution

Spanning set of the whole space

Example 22

Determine whether $P_1 = 1 + X + X^2$, $P_2 = 3 + X$, $P_3 = 5 - X + 4X^2$, and $P_4 = -2 - 2X + 2X^2$ span $\mathbb{P}_2$.

Solution

Standard spanning sets

Note:

- The standard spanning set for $\mathbb{R}^2$ is e_1, e_2, where

$$e_1 = (1, 0) \text{ and } e_2 = (0, 1)$$

- The standard spanning set for $\mathbb{R}^3$ is e_1, e_2, e_3, where

$$e_1 = (1, 0, 0),\ e_2 = (0, 1, 0) \text{ and } e_3 = (0, 0, 1)$$

- The standard spanning set for $\mathbb{R}^n$ is $e_1, e_2, e_3, \ldots, e_n$, where

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, \ldots, 0),\ e_3 = (0, 0, 1, 0, \ldots, 0) \text{ and }$$
$$e_n = (0, 0, \ldots, 1)$$

- The standard spanning set for $\mathbb{P}_2$ is $1, X, X^2$.
- The standard spanning set for $\mathbb{P}_n$ is $1, X, X^2, \ldots, X^n$.
- The standard spanning set for $\mathrm{Mat}(2, 2, \mathbb{R})$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

Spanning set of the subspace

Example 23

Find a spanning set for the following subspace
Let m be a fixed real number. Consider the subset of $V = \mathbb{R}^2$

$$W := \left\{ \begin{pmatrix} x \\ mx \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

Spanning set of the subspace

Example 24

Find a spanning set for the following subspace

$$W := \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R}, c = a - b \right\}$$

Spanning set of the subspace

Example 25

Find a spanning set for the following subspace

$$W := \left\{ A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^T = -A \right\}$$

Spanning set of the subspace

Example 26

Find a spanning set for the following subspace

$$W := \left\{a_0 + a_1X + a_2X^2 + a_3X^3 \mid a_1 = a_2\right\}$$

Null Space

Theorem 27

Let $A \in Mat(m, n, \mathbb{R})$ be a $m \times n$ matrix. The subset W of $\mathbb{R}^n$ defined by

$$W = \{\mathbf{x} | \, A\mathbf{x} = \mathbf{0}\}$$

is a subspace of $\mathbb{R}^n$.

It is called the **null space** of A, denoted by $\text{Nul}(A)$ and it is consisting of the solutions to the equation $A\mathbf{x} = \mathbf{0}$.

Example 28

Determine whether the $\mathbf{w} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ is in the null space of

$$A = \begin{pmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{pmatrix}$$

Spanning set of the subspace

Example 29

Find a spanning set for the null space of

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 6 & -9 & 3 \\ -4 & 6 & -2 \end{pmatrix}$$

Spanning set of the subspace

Example 30

Find a spanning set for the null space of

$$A = \begin{pmatrix} 1 & 4 & 8 \\ 2 & 5 & 6 \\ 3 & 1 & -4 \end{pmatrix}$$

Do HOMEWORK 1

Equality of spanning sets

Theorem 31

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ *and* $S' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ *be two sets of vectors. Then,*

$$span\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = span\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$$

$$\iff$$

each $\mathbf{v}_i$ *is a linear combination of* $\mathbf{w}_i$ *and each* $\mathbf{w}_i$ *is a linear combination of the* $\mathbf{v}_i$

Example 32

Show that

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}=\operatorname{span}\left\{\begin{pmatrix}4\\3\\0\end{pmatrix},\begin{pmatrix}3\\2\\0\end{pmatrix}\right\}$$