Section 4.7
Row and Column Spaces

Dr. Abdulla Eid

College of Science

MATHS 211: Linear Algebra

Goal:

1. Define the Row and Column Spaces of a matrix.
2. Find basis for the row and column spaces of a matrix.
3. Relation between column space and null space.

1 - Define row space and column space

Example 1

Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -6 \\ 1 & 4 & 2 & 7 \end{pmatrix}$$

1. Extract from A vectors in $\mathbb{R}^4$.
2. Extract from A subspace in $\mathbb{R}^4$.
3. Extract from A vectors in $\mathbb{R}^3$.
4. Extract from A subspace in $\mathbb{R}^3$.

Example 2

Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 & 9 \\ 3 & 5 & 7 & -6 & 4 \end{pmatrix}$$

1. Extract from A vectors in $\mathbb{R}^5$.
2. Extract from A subspace in $\mathbb{R}^5$.
3. Extract from A vectors in $\mathbb{R}^2$.
4. Extract from A subspace in $\mathbb{R}^2$.

Definition 3

Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the **row space** of A, denoted by $\text{Row}(A)$.

Definition 4

The subspace of $\mathbb{R}^m$ spanned by the columns of A is called the **column space** of A and is denoted by $\text{Col}(A)$.

2 - Finding basis for the column and row space of a matrix

1. Reduce A into $RREF$ matrix B.
2. The basis for the row space of A are those rows in A (or in B) that correspond to the pivot rows in B.
3. The basis for the column space of A are those columns in A that correspond to the pivot columns in B.

Example 5

Find a basis for the row space and column space of the matrix

$$A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$$

Example 6

Find a basis for the row space, column, and null space of the matrix

$$A = \begin{pmatrix} 1 & -2 & 10 \\ 2 & -3 & 18 \\ 0 & -7 & 14 \end{pmatrix}$$

Example 7

Find a basis for the row space and column space of the matrix

$$A = \begin{pmatrix} 3 & 4 \\ -6 & 10 \end{pmatrix}$$

Example 8

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors:
$\mathbf{v}_1 = (2, 4, -2, 3), \mathbf{v}_2 = (-2, -2, 2, -4), \mathbf{v}_3 = (1, 3, -1, 1)$

3 - Relation between column space and null space

Example 9

Express the product as a linear combination of the columns of A.

1. $\begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
2. $\begin{pmatrix} 5 & 2 & 6 \\ 1 & -1 & 3 \\ 0 & 1 & 7 \\ 1 & 7 & 3 \\ 4 & -1 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$

3 - Relation between column space and null space

Recall:

$$\text{Nul}(A) := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$$

and by the above

$$\text{Col}(A) := \{b \in \mathbb{R}^m \mid Ax = b, \text{ for some } x \in \mathbb{R}^n\}$$

Example 10

Determine whether b is in the column space of A or not.

$$A = \begin{pmatrix} 0 & 1 & 4 \\ 2 & 1 & 1 \\ 2 & 2 & 5 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$