# Section 6.3
# Orthogonal and orthonormal basis


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MATHS 211: Linear Algebra

Goal:

1. Orthogonal and orthonormal basis.
2. Coordinates relatives to orthonormal basis.
3. Orthogonal Projection.
4. The Gram–Schmidt Process.

# Normalizing Procedure

Goal: To create a unit vector $v'$ from a given vector $v$.

$$v' = \frac{1}{||v||}v$$

Check that $v'$ indeed is a unit vector!

# Orthogonal Set

## Definition 1

A set of vectors $\{\mathbf{v}_1\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called **orthogonal set** if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{0}$, for all $i \neq j$.
A set if called **orthonormal basis** if it is orthogonal and each vector is a unit vector.

## Example 2

Verify that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ is an orthogonal set.
Find an orthonormal set.

# Orthogonality implies linearly independent

## Theorem 3

*If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set, then $S$ is linearly independent.*



## Definition 4

A basis consisting of orthogonal vectors is called **orthogonal basis**. Similarly, a basis consisting of orthonormal vectors is called **orthonormal basis**.

# Relative coordinates to orthonormal basis

## Theorem 5

*(a) If* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an orthogonal basis then*

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

*with* $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{||\mathbf{v}_i||^2}$.
*(b) In case* $S$ *is orthonormal basis, then* $c_i = \langle \mathbf{u}, \mathbf{v}_i \rangle$.

Proof: Consider $||\mathbf{u} + \mathbf{v}||^2$.

## Example 6

Verify that the vectors $\mathbf{v}_1 = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an orthogonal basis. Then write each of the following vectors as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(a) $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

(b) $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

(c) $\mathbf{u} = \begin{pmatrix} \frac{1}{7} \\ \frac{-3}{7} \\ \frac{5}{7} \end{pmatrix}$

# Orthogonal Projection

## Theorem 7

*Let $W$ be a subspace of $V$, then each vector $\mathbf{v} \in V$ can be written in exactly one way as*

$$\mathbf{v} = \mathbf{w} + \hat{\mathbf{w}}$$

*where $\mathbf{w} \in W$ and $\hat{w} \in W^{\perp}$.*



The vector $\mathbf{w}$ above is called **orthogonal projection of u** on $W$ and denoted by $\mathbf{w} = \text{proj}_W \mathbf{u}$. The vector $\hat{\mathbf{w}}$ above is called **orthogonal projection of u** on $W^{\perp}$ and denoted by $\mathbf{w} = \text{proj}_W^{\perp} \mathbf{u}$.

# How to calculate $\text{proj}_W\mathbf{u}$

First we find an *orthogonal* basis for $W$, say $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then,

$$\text{proj}_W\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

with $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{||\mathbf{v}_i||^2}$

# Calculate the projections

## Example 8

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ be a basis for a subspace $W$. Find $\text{proj}_W \mathbf{u}$,

where $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}$.

# Calculate the projections

## Example 9

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ be a basis for a subspace $W$. Find $\text{proj}_W \mathbf{u}$, where $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}$.

# Creating orthogonal basis from any basis

## Definition 10

If $W$ is a subspace of an inner vector space $V$, then the set of all vectors in $V$ that are orthogonal to every vector in $W$ is called the **orthogonal complement** of $W$ and is denoted by $W^{\perp}$.

$$W^{\perp} := \{\hat{w} \in V \mid \langle \hat{w}, w \rangle = 0, \text{ for all } w \in W\}$$



## Theorem 11

1. *$W^{\perp}$ is a subspace of $W$.*
2. $W \cap W^{\perp} = \{\mathbf{0}\}$
3. $(W^{\perp})^{\perp}$

# Row space and null space are orthogonal

## Example 12

Let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where,

$$\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 4 \\ -5 \\ 13 \end{pmatrix}$$

# Row space and null space are orthogonal

## Example 13

Let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where,

$$\mathbf{w}_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} -1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 4 \\ 2 \\ 3 \\ -3 \end{pmatrix}$$