Section 8.1
Linear Transformation
Part 2: Null and Range of a linear transformation

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MATHS 211: Linear Algebra

Goal:

1. Define Linear Transformations.
2. Examples of Linear Transformations.
3. Finding the linear transformation from a basis.
4. Kernel and Range of a linear transformations.
5. Properties and of Kernel and Images.
6. Rank and Nullity of a linear transformation.

Definition 1

If $T : V \to W$ be a linear transformation. The set of all vectors in V that T maps into zero is called the **kernel** of T and denoted by $\ker(T)$.the set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

$$\ker(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$$

$$R(T) := \{\mathbf{w} \in W \mid T(\mathbf{v}) = \mathbf{w}, \text{ for some } \mathbf{v} \in V\}$$

Example 2

Define a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} 3x + y \\ y \end{pmatrix}\right)$$

(a) Find a basis for the null space and its dimension.
(b) Give a description of the range of T.
(c) Find a basis for the range space and its dimension.

Example 3

Define a mapping $T : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \left(\begin{pmatrix} x+y-z+w \\ 2x+y+4z+w \\ 3x+y+9z \end{pmatrix}\right)$$

(a) Find a basis for the null space and its dimension.
(b) Give a description of the range of T.
(c) Find a basis for the range space and its dimension.

Example 4

Define a mapping $T : \mathbb{P}_2 \to \mathbb{R}$ by

$$T\left(p(X)\right) = \left(P(0)\right)$$

(a) Find a basis for the null space and its dimension.
(b) Give a description of the range of T.
(c) Find a basis for the range space and its dimension.

Example 5

Define a mapping $T : \mathbb{P}_3 \to \mathbb{P}_3$ by

$$T\left(p(X)\right) = \left(P''(X) + p'(X) + p(0)\right)$$

(a) Find a basis for the null space and its dimension.
(b) Give a description of the range of T.
(c) Find a basis for the range space and its dimension.

Matrix Transformation

Example 6

Let A be any $m \times n$ matrix. Define the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

$\ker(T_A)$ is the Null space of A while $R(T_A)$ is the Column space of A.

Zero Transformation

Example 7

Let V be any vector space. Define the linear transformation $T : V \to W$ by

$$T(\mathbf{x}) = \mathbf{0}_W$$

Identity Operator

Example 8

Let V be any vector space. Define the linear transformation (operator) $T : V \to V$ by

$$T(\mathbf{x}) = \mathbf{x}$$

Dilation and Contraction Operators

Example 9

Let V be any vector space. Define the linear operator $T_k : V \to V$ by

$$T_k(\mathbf{x}) = k\mathbf{x}$$

If $0 < k < 1$, then T_k is called the **contraction** of V with factor k and if $k > 1$, it is called the **dilation** of V with factor k.

A Linear Transformation from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$

Example 10

Define the linear operator $T : \mathbb{P}_n \to \mathbb{P}_n$ by

$$T(p(X)) = XP(X)$$

$$T(c_0 + c_1X + c_2X^2 + \cdots + c_nX^n) = c_0X + c_1X^2 + c_2X^3 + \cdots + c_nX^{n+1}$$

Theorem 11

Let $T : V \to W$ be a linear transformation. $\ker(T)$ *and* $R(A)$ *are both subspaces of* V *and* W.

1. Dimension of the $\ker(T)$ is called the **nullity** of T.
2. Dimension of the $R(A)$ is called the **rank** of T.

Theorem 12

Dimension Theorem Let $T : V \to W$ be a linear transformation, then

$$rank(T) + Nullity(T) = \dim(V) = n$$

Example 13

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y) = (x - 3y, -2x + 6y)$$

Which of the following vectors are in $R(T)$?
$(1, -2)$, $(3, 1)$, $(-2, 4)$ Which of the following vectors are in $ker(T)$?
$(1, -3)$, $(3, 1)$, $(-6, -2)$ Find a basis for the $\ker(T)$? Find a basis for the $R(T)$? Verify the formula in the dimension theorem?

Example 14

Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation given by

$$T(p(X)) = (X+1)p(X)$$

Which of the following vectors are in $R(T)$?
$X + X^2$, $1 + X$, $3 - X^2$ Which of the following vectors are in $\ker(T)$?
X^2, 0, $X + 1$ Find a basis for the $\ker(T)$? Find a basis for the $R(T)$?
Verify the formula in the dimension theorem?